PHYSICS NOTES

ROLL NO :- ________

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CONTENTS

1. Nature of the Physical World and Measurement

The history of humans reveals that they have been making continuous and serious attempts to understand the world around them. The repetition of day and night, cycle of seasons, volcanoes, rainbows, eclipses and the starry night sky have always been a source of wonder and subject of thought. The inquiring mind of humans always tried to understand the natural phenomena by observing the environment carefully. This pursuit of understanding nature led us to today's modern science and technology.

1.1 Physics

The word science comes from a Latin word "scientia" which means 'to know'. Science is nothing but the knowledge gained through the systematic observations and experiments. Scientific methods include the systematic observations, reasoning, modelling and theoretical prediction. Science has many disciplines, physics being one of them. The word physics has its origin in a Greek word meaning 'nature'. Physics is the most basic science, which deals with the study of nature and natural phenomena. Understanding science begins with understanding physics. With every passing day, physics has brought to us deeper levels of understanding of nature.

Physics is an empirical study. Everything we know about physical world and about the principles that govern its behaviour has been learned through observations of the phenomena of nature. The ultimate test of any physical theory is its agreement with observations and measurements of physical phenomena. Thus physics is inherently a science of measurement.

1.1.1 Scope of Physics

The scope of physics can be understood if one looks at its various sub-disciplines such as mechanics, optics, heat and thermodynamics, electrodynamics, atomic physics, nuclear physics, etc.

Mechanics deals with motion of particles and general systems of particles. The working of telescopes, colours of thin films are the topics dealt in optics. Heat and thermodynamics deals with the pressure - volume changes that take place in a gas when its temperature changes, working of refrigerator, etc. The phenomena of charged particles and magnetic bodies are dealt in electrodynamics. The magnetic field around a current carrying conductor, propagation of radio waves etc. are the areas where electrodynamics provide an answer. Atomic and nuclear physics deals with the constitution and structure of matter, interaction of atoms and nuclei with electrons, photons and other elementary particles.

Foundation of physics enables us to appreciate and enjoy things and happenings around us. The laws of physics help us to understand and comprehend the cause-effect relationships in what we observe. This makes a complex problem to appear pretty simple.

Physics is exciting in many ways. To some, the excitement comes from the fact that certain basic concepts and laws can explain a range of phenomena. For some others, the thrill lies in carrying out new experiments to unravel the secrets of nature. Applied physics is even more interesting. Transforming laws and theories into useful applications require great ingenuity and persistent effort.

1.1.2 Physics, Technology and Society

Technology is the application of the doctrines in physics for practical purposes. The invention of steam engine had a great impact on human civilization. Till 1933, Rutherford did not believe that energy could be tapped from atoms. But in 1938, Hann and Meitner discovered neutron-induced fission reaction of uranium. This is the basis of nuclear weapons and nuclear reactors. The contribution of physics in the development of alternative resources of energy is significant. We are consuming the fossil fuels at such a very fast rate that there is an urgent need to discover new sources of energy which are cheap. Production of electricity from solar energy and geothermal energy is a reality now, but we have a long way to go. Another example of physics giving rise to technology is the integrated chip, popularly called as IC. The development of newer ICs and faster processors made the computer industry to grow leaps and bounds in the last two decades. Computers have become affordable now due to improved production techniques and low production costs.

The legitimate purpose of technology is to serve poeple. Our society is becoming more and more science-oriented. We can become better members of society if we develop an understanding of the basic laws of physics.

1.2 Forces of nature

Sir Issac Newton was the first one to give an exact definition for force.

"Force is the external agency applied on a body to change its state of rest and motion".

There are four basic forces in nature. They are gravitational force, electromagnetic force, strong nuclear force and weak nuclear force.

Gravitational force

It is the force between any two objects in the universe. It is an attractive force by virtue of their masses. By Newton's law of gravitation, the gravitational force is directly proportional to the product of the masses and inversely proportional to the square of the distance between them. Gravitational force is the weakest force among the fundamental forces of nature but has the greatest large–scale impact on the universe. Unlike the other forces, gravity works universally on all matter and energy, and is universally attractive.

Electromagnetic force

It is the force between charged particles such as the force between two electrons, or the force between two current carrying wires. It is attractive for unlike charges and repulsive for like charges. The electromagnetic force obeys inverse square law. It is very strong compared to the gravitational force. It is the combination of electrostatic and magnetic forces.

Strong nuclear force

It is the strongest of all the basic forces of nature. It, however, has the shortest range, of the order of 10^{-15} m. This force holds the protons and neutrons together in the nucleus of an atom.

Weak nuclear force

Weak nuclear force is important in certain types of nuclear process such as β-decay. This force is not as weak as the gravitational force.

1.3 Measurement

Physics can also be defined as the branch of science dealing with the study of properties of materials. To understand the properties of materials, measurement of physical quantities such as length, mass, time etc., are involved. The uniqueness of physics lies in the measurement of these physical quantities.

1.3.1 Fundamental quantities and derived quantities

Physical quantities can be classified into two namely, fundamental quantities and derived quantities. *Fundamental quantities are quantities which cannot be expressed in terms of any other physical quantity.* For example, quantities like length, mass, time, temperature are fundamental quantities. *Quantities that can be expressed in terms of fundamental quantities are called derived quantities.* Area, volume, density etc. are examples for derived quantities.

1.3.2 Unit

To measure a quantity, we always compare it with some reference standard. To say that a rope is 10 metres long is to say that it is 10 times as long as an object whose length is defined as 1 metre. Such a standard is called a unit of the quantity.

Therefore, *unit of a physical quantity is defined as the established standard used for comparison of the given physical quantity.*

The units in which the fundamental quantities are measured are called fundamental units and the units used to measure derived quantities are called derived units.

1.3.3 System International de Units (SI system of units)

In earlier days, many system of units were followed to measure physical quantities. The British system of foot–pound–second or fps system, the Gaussian system of centimetre – gram – second or cgs system, the metre–kilogram – second or the mks system were the three systems commonly followed. To bring uniformity, the General Conference on Weights and Measures in the year 1960, accepted the SI system of units. This system is essentially a modification over mks system and is, therefore rationalised mksA (metre kilogram second ampere) system. This rationalisation was essential to obtain the units of all the physical quantities in physics.

In the SI system of units there are seven fundamental quantities and two supplementary quantities. They are presented in Table 1.1.

Table 1.1 SI system of units

Physical quantity	Unit	Symbol
Fundamental quantities		
Length	metre	m
Mass	kilogram	kg
Time	second	s
Electric current	ampere	A
Temperature	kelvin	K
Luminous intensity	candela	cd
Amount of substance	mole	mol
Supplementary quantities		
Plane angle	radian	rad
Solid angle	steradian	sr

1.3.4 Uniqueness of SI system

The SI system is logically far superior to all other systems. The SI units have certain special features which make them more convenient in practice. Permanence and reproduceability are the two important characteristics of any unit standard. The SI standards do not vary with time as they are based on the properties of atoms. Further *SI system of units are coherent system of units, in which the units of derived quantities are obtained as multiples or submultiples of certain basic units.* Table 1.2 lists some of the derived quantities and their units.

Table 1.2 Derived quantities and their units

Physical Quantity	Expression	Unit
Area	length × breadth	m^2
Volume	area × height	m^3
Velocity	displacement/ time	$m\ s^{-1}$
Acceleration	velocity / time	$m\ s^{-2}$
Angular velocity	angular displacement / time	$rad\ s^{-1}$
Angular acceleration	angular velocity / time	$rad\ s^{-2}$
Density	mass / volume	$kg\ m^{-3}$
Momentum	mass × velocity	$kg\ m\ s^{-1}$
Moment of intertia	mass × (distance)^2	$kg\ m^2$
Force	mass × acceleration	$kg\ m\ s^{-2}$ or N
Pressure	force / area	$N\ m^{-2}$ or Pa
Energy (work)	force × distance	N m or J
Impulse	force × time	N s
Surface tension	force / length	$N\ m^{-1}$
Moment of force (torque)	force × distance	N m
Electric charge	current × time	A s
Current density	current / area	$A\ m^{-2}$
Magnetic induction	force / (current × length)	$N\ A^{-1}\ m^{-1}$

1.3.5 SI standards

Length

Length is defined as the distance between two points. The SI unit of length is metre.

One standard metre is equal to 1 650 763.73 wavelengths of the orange – red light emitted by the individual atoms of krypton – 86 in a krypton discharge lamp.

Mass

Mass is the quantity of matter contained in a body. It is independent of temperature and pressure. It does not vary from place to place. The SI unit of mass is kilogram.

The kilogram is equal to the mass of the international prototype of the kilogram (a plantinum – iridium alloy cylinder) kept at the International Bureau of Weights and Measures at Sevres, near Paris, France.

An atomic standard of mass has not yet been adopted because it is not yet possible to measure masses on an atomic scale with as much precision as on a macroscopic scale.

Time

Until 1960 the standard of time was based on the mean solar day, the time interval between successive passages of the sun at its highest point across the meridian. It is averaged over an year. In 1967, an atomic standard was adopted for second, the SI unit of time.

One standard second is defined as the time taken for 9 192 631 770 periods of the radiation corresponding to unperturbed transition between hyperfine levels of the ground state of cesium – 133 atom. Atomic clocks are based on this. In atomic clocks, an error of one second occurs only in 5000 years.

Ampere

The ampere is the constant current which, flowing through two straight parallel infinitely long conductors of negligible cross-section, and placed in vacuum 1 m apart, would produce between the conductors a force of 2×10^{-7} *newton per unit length of the conductors.*

Kelvin

The Kelvin is the fraction of $\frac{1}{273.16}$ *of the thermodynamic temperature of the triple point of water*.*

Candela

The candela is the luminous intensity in a given direction due to a

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** Triple point of water is the temperature at which saturated water vapour, pure water and melting ice are all in equilibrium. The triple point temperature of water is 273.16 K.*

source, which emits monochromatic radiation of frequency 540×10^{12} Hz and of which the radiant intensity in that direction is $\frac{1}{683}$ watt per steradian.

Mole

The mole is the amount of substance which contains as many elementary entities as there are atoms in 0.012 kg of carbon-12.

1.3.6 Rules and conventions for writing SI units and their symbols

1. The units named after scientists are not written with a capital initial letter.

For example : newton, henry, watt

2. The symbols of the units named after scientist should be written by a capital letter.

For example : N for newton, H for henry, W for watt

3. Small letters are used as symbols for units not derived from a proper name.

For example : m for metre, kg for kilogram

4. No full stop or other punctuation marks should be used within or at the end of symbols.

For example : 50 m and not as 50 m.

5. The symbols of the units do not take plural form.

For example : 10 kg not as 10 kgs

6. When temperature is expressed in kelvin, the degree sign is omitted.

For example : 273 K not as 273^{o} K

(If expressed in Celsius scale, degree sign is to be included. For example 100^{o} C and not 100 C)

7. Use of solidus is recommended only for indicating a division of one letter unit symbol by another unit symbol. Not more than one solidus is used.

For example : m s^{-1} or m / s, J / K mol or J K^{-1} mol^{-1} but not J / K / mol.

8. Some space is always to be left between the number and the symbol of the unit and also between the symbols for compound units such as force, momentum, etc.

For example, it is not correct to write 2.3m. The correct representation is 2.3 m; kg m s^{-2} and not as $kgms^{-2}$.

9. Only accepted symbols should be used.

For example : ampere is represented as A and not as amp. or am ; second is represented as s and not as sec.

10. Numerical value of any physical quantity should be expressed in scientific notation.

For an example, density of mercury is 1.36×10^4 kg m^{-3} and not as 13600 kg m^{-3}.

1.4 Expressing larger and smaller physical quantities

Once the fundamental units are defined, it is easier to express larger and smaller units of the same physical quantity. In the metric (SI) system these are related to the fundamental unit in multiples of 10 or 1/10. Thus 1 km is 1000 m and 1 mm is 1/1000 metre. Table 1.3 lists the standard SI prefixes, their meanings and abbreviations.

Table 1.3 Prefixes for power of ten

Power of ten	Prefix	Abbreviation
10^{-15}	femto	f
10^{-12}	pico	p
10^{-9}	nano	n
10^{-6}	micro	μ
10^{-3}	milli	m
10^{-2}	centi	c
10^{-1}	deci	d
10^{1}	deca	da
10^{2}	hecto	h
10^{3}	kilo	k
10^{6}	mega	M
10^{9}	giga	G
10^{12}	tera	T
10^{15}	peta	P

In order to measure very large distances, the following units are used.

(i) Light year

Light year is the distance travelled by light in one year in vacuum.

Distance travelled = velocity of light × 1 year

$\therefore$ 1 light year $= 3 \times 10^8 \text{ m s}^{-1} \times 1$ year (in seconds)

$= 3 \times 10^8 \times 365.25 \times 24 \times 60 \times 60$

$= 9.467 \times 10^{15}$ m

1 light year $= 9.467 \times 10^{15}$ *m*

(ii) Astronomical unit

Astronomical unit is the mean distance of the centre of the Sun from the centre of the Earth.

1 Astronomical unit (AU) $= 1.496 \times 10^{11}$ *m*

1.5 Determination of distance

For measuring large distances such as the distance of moon or a planet from the Earth, special methods are adopted. Radio-echo method, laser pulse method and parallax method are used to determine very large distances.

Laser pulse method

The distance of moon from the Earth can be determined using laser pulses. The laser pulses are beamed towards the moon from a powerful transmitter. These pulses are reflected back from the surface of the moon. The time interval between sending and receiving of the signal is determined very accurately.

If t is the time interval and c the velocity of the laser pulses, then the distance of the moon from the Earth is $d = \frac{ct}{2}$.

1.6 Determination of mass

The conventional method of finding the mass of a body in the laboratory is by physical balance. The mass can be determined to an accuracy of 1 mg. Now–a–days, digital balances are used to find the mass very accurately. The advantage of digital balance is that the mass of the object is determined at once.

1.7 Measurement of time

We need a clock to measure any time interval. Atomic clocks provide better standard for time. Some techniques to measure time interval are given below.

Quartz clocks

The *piezo–electric property** of a crystal is the principle of quartz clock. These clocks have an accuracy of one second in every 10^9 seconds.

Atomic clocks

These clocks make use of periodic vibration taking place within the atom. Atomic clocks have an accuracy of 1 part in 10^{13} seconds.

1.8 Accuracy and precision of measuring instruments

All measurements are made with the help of instruments. The accuracy to which a measurement is made depends on several factors. For example, if length is measured using a metre scale which has graduations at 1 mm interval then all readings are good only upto this value. *The error in the use of any instrument is normally taken to be half of the smallest division on the scale of the instrument. Such an error is called instrumental error.* In the case of a metre scale, this error is about 0.5 mm.

Physical quantities obtained from experimental observation always have some uncertainity. Measurements can never be made with absolute precision. Precision of a number is often indicated by following it with $\pm$ symbol and a second number indicating the maximum error likely.

For example, if the length of a steel rod = 56.47 $\pm$ 3 mm then the true length is unlikely to be less than 56.44 mm or greater than 56.50 mm. *If the error in the measured value is expressed in fraction, it is called fractional error and if expressed in percentage it is called percentage error.* For example, a resistor labelled "470 Ω, 10%" probably has a true resistance differing not more than 10% from 470 Ω. So the true value lies between 423 Ω and 517 Ω.

1.8.1 Significant figures

The digits which tell us the number of units we are reasonably sure of having counted in making a measurement are called significant figures. Or in other words, *the number of meaningful digits in a number is called the number of significant figures. A choice of change of different units does not change the number of significant digits or figures in a measurement.*

** When pressure is applied along a particular axis of a crystal, an electric potential difference is developed in a perpendicular axis.*

For example, 2.868 cm has four significant figures. But in different units, the same can be written as 0.02868 m or 28.68 mm or 28680 μm. All these numbers have the same four significant figures.

From the above example, we have the following rules.

i) All the non–zero digits in a number are significant.

ii) All the zeroes between two non–zeroes digits are significant, irrespective of the decimal point.

iii) If the number is less than 1, the zeroes on the right of decimal point but to the left of the first non–zero digit are not significant. (In 0.02868 the underlined zeroes are not significant).

iv) The zeroes at the end without a decimal point are not significant. (In 23080 μm, the trailing zero is not significant).

v) The trailing zeroes in a number with a decimal point are significant. (The number 0.07100 has four significant digits).

Examples

i) 30700 has three significant figures.

ii) 132.73 has five significant figures.

iii) 0.00345 has three and

iv) 40.00 has four significant figures.

1.8.2 Rounding off

Calculators are widely used now–a–days to do the calculations. The result given by a calculator has too many figures. In no case the result should have more significant figures than the figures involved in the data used for calculation. The result of calculation with number containing more than one uncertain digit, should be rounded off. The technique of rounding off is followed in applied areas of science.

A number 1.876 rounded off to three significant digits is 1.88 while the number 1.872 would be 1.87. The rule is that if the insignificant digit (underlined) is more than 5, the preceeding digit is raised by 1, and is left unchanged if the former is less than 5.

If the number is 2.845, the insignificant digit is 5. In this case, the convention is that if the preceeding digit is even, the insignificant digit is simply dropped and, if it is odd, the preceeding digit is raised by 1. Following this, 2.845 is rounded off to 2.84 where as 2.815 is rounded off to 2.82.

Examples

1. Add 17.35 kg, 25.8 kg and 9.423 kg.

Of the three measurements given, 25.8 kg is the least accurately known.

$\therefore$ 17.35 + 25.8 + 9.423 = 52.573 kg

Correct to three significant figures, 52.573 kg is written as 52.6 kg

2. Multiply 3.8 and 0.125 with due regard to significant figures.

$3.8 \times 0.125 = 0.475$

The least number of significant figure in the given quantities is 2. Therefore the result should have only two significant figures.

$\therefore 3.8 \times 0.125 = 0.475 = 0.48$

1.8.3 Errors in Measurement

The uncertainity in the measurement of a physical quantity is called error. It is the difference between the true value and the measured value of the physical quantity. Errors may be classified into many categories.

(i) Constant errors

It is the same error repeated every time in a series of observations. Constant error is due to faulty calibration of the scale in the measuring instrument. In order to minimise constant error, measurements are made by different possible methods and the mean value so obtained is regarded as the true value.

(ii) Systematic errors

These are errors which occur due to a certain pattern or system. These errors can be minimised by identifying the source of error. Instrumental errors, personal errors due to individual traits and errors due to external sources are some of the systematic errors.

(iii) Gross errors

Gross errors arise due to one or more than one of the following reasons.

(1) Improper setting of the instrument.

(2) Wrong recordings of the observation.

(3) Not taking into account sources of error and precautions.

(4) Usage of wrong values in the calculation.

Gross errros can be minimised only if the observer is very careful in his observations and sincere in his approach.

(iv) Random errors

It is very common that repeated measurements of a quantity give values which are slightly different from each other. *These errors have no set pattern and occur in a random manner.* Hence they are called random errors. They can be minimised by repeating the measurements many times and taking the arithmetic mean of all the values as the correct reading.

The most common way of expressing an error is percentage error. If the accuracy in measuring a quantity x is Δx, then the percentage error in x is given by $\frac{\Delta x}{x} \times 100$ %.

1.9 Dimensional Analysis

Dimensions of a physical quantity are the powers to which the fundamental quantities must be raised.

$$\text{We know that velocity} = \frac{\text{displacement}}{\text{time}}$$

$$= \frac{[L]}{[T]}$$

$$= [M^oL^1T^{-1}]$$

where [M], [L] and [T] are the dimensions of the fundamental quantities mass, length and time respectively.

Therefore velocity has zero dimension in mass, one dimension in length and –1 dimension in time. Thus the dimensional formula for velocity is $[M^oL^1T^{-1}]$ or simply $[LT^{-1}]$.The dimensions of fundamental quantities are given in Table 1.4 and the dimensions of some derived quantities are given in Table 1.5

Table 1.4 Dimensions of fundamental quantities

Fundamental quantity	Dimension
Length	L
Mass	M
Time	T
Temperature	K
Electric current	A
Luminous intensity	cd
Amount of subtance	mol

Table 1.5 Dimensional formulae of some derived quantities

Physical quantity	Expression	Dimensional formula
Area	length × breadth	$[L^2]$
Density	mass / volume	$[ML^{-3}]$
Acceleration	velocity / time	$[LT^{-2}]$
Momentum	mass × velocity	$[MLT^{-1}]$
Force	mass × acceleration	$[MLT^{-2}]$
Work	force × distance	$[ML^2T^{-2}]$
Power	work / time	$[ML^2T^{-3}]$
Energy	work	$[ML^2T^{-2}]$
Impulse	force × time	$[MLT^{-1}]$
Radius of gyration	distance	$[L]$
Pressure	force / area	$[ML^{-1}T^{-2}]$
Surface tension	force / length	$[MT^{-2}]$
Frequency	1 / time period	$[T^{-1}]$
Tension	force	$[MLT^{-2}]$
Moment of force (or torque)	force × distance	$[ML^2T^{-2}]$
Angular velocity	angular displacement / time	$[T^{-1}]$
Stress	force / area	$[ML^{-1}T^{-2}]$
Heat	energy	$[ML^2T^{-2}]$
Heat capacity	heat energy/ temperature	$[ML^2T^{-2}K^{-1}]$
Charge	current × time	$[AT]$
Faraday constant	Avogadro constant × elementary charge	$[AT\ mol^{-1}]$
Magnetic induction	force / (current × length)	$[MT^{-2}\ A^{-1}]$

Dimensional quantities

Constants which possess dimensions are called dimensional constants. Planck's constant, universal gravitational constant are dimensional constants.

Dimensional variables are those physical quantities which possess dimensions but do not have a fixed value. Example – velocity, force, etc.

Dimensionless quantities

There are certain quantities which do not possess dimensions. They are called dimensionless quantities. Examples are strain, angle, specific gravity, etc. They are dimensionless as they are the ratio of two quantities having the same dimensional formula.

Principle of homogeneity of dimensions

An equation is dimensionally correct if the dimensions of the various terms on either side of the equation are the same. This is called the principle of homogeneity of dimensions. This principle is based on the fact that two quantities of the same dimension only can be added up, the resulting quantity also possessing the same dimension.

The equation A + B = C is valid only if the dimensions of A, B and C are the same.

1.9.1 Uses of dimensional analysis

The method of dimensional analysis is used to

(i) convert a physical quantity from one system of units to another.

(ii) check the dimensional correctness of a given equation.

(iii) establish a relationship between different physical quantities in an equation.

(i) To convert a physical quantity from one system of units to another

Given the value of G in cgs system is $6.67 \times 10^{-8} dyne\, cm^2\, g^{-2}$. Calculate its value in SI units.

In cgs system	In SI system
$G_{cgs} = 6.67 \times 10^{-8}$	$G = ?$
$M_1 = 1g$	$M_2 = 1\ kg$
$L_1 = 1\ cm$	$L_2 = 1m$
$T_1 = 1s$	$T_2 = 1s$

The dimensional formula for gravitational constant is $\left[M^{-1}L^{3}T^{-2}\right]$.

In *cgs* system, dimensional formula for G is $\left[M_1^x\ L_1^y\ T_1^z\right]$

In SI system, dimensional formula for G is $\left[M_2^x\ L_2^y\ T_2^z\right]$

Here $x = -1$, $y = 3$, $z = -2$

$$\therefore \quad G\left[M_2{}^xL_2{}^yT_2{}^z\right] = G_{cgs}\left[M_1{}^xL_1{}^yT_1{}^z\right]$$

$$\text{or} \quad G = G_{cgs}\left[\frac{M_1}{M_2}\right]^x\left[\frac{L_1}{L_2}\right]^y\left[\frac{T_1}{T_2}\right]^z$$

$$= 6.67 \times 10^{-8}\left[\frac{1\ g}{1\ kg}\right]^{-1}\left[\frac{1\ cm}{1\ m}\right]^3\left[\frac{1\ s}{1\ s}\right]^{-2}$$

$$= 6.67 \times 10^{-8}\left[\frac{1\ g}{1000\ g}\right]^{-1}\left[\frac{1\ cm}{100\ cm}\right]^3[1]^{-2}$$

$$= 6.67 \times 10^{-11}$$

$\therefore$ In SI units,

$G = 6.67 \times 10^{-11}\ N\ m^2\ kg^{-2}$

(ii) To check the dimensional correctness of a given equation

Let us take the equation of motion

$$s = ut + (½)at^2$$

Applying dimensions on both sides,

$$[L] = [LT^{-1}]\ [T] + [LT^{-2}]\ [T^2]$$

(½ is a constant having no dimension)

$$[L] = [L] + [L]$$

As the dimensions on both sides are the same, the equation is dimensionally correct.

(iii) To establish a relationship between the physical quantities in an equation

Let us find an expression for the time period *T* of a simple pendulum. The time period *T* may depend upon (i) mass *m* of the bob (ii) length *l* of the pendulum and (iii) acceleration due to gravity *g* at the place where the pendulum is suspended.

(i.e) $T \propto m^x l^y g^z$

or $T = k\, m^x l^y g^z$ ...(1)

where k is a dimensionless constant of propotionality. Rewriting equation (1) with dimensions,

$$[T^1] = [M^x]\,[L^y]\,[LT^{-2}]^z$$

$$[T^1] = [M^x L^{y+z} T^{-2z}]$$

Comparing the powers of M, L and T on both sides

$$x = 0,\; y + z = 0 \text{ and } -2z = 1$$

Solving for x, y and z, $x = 0$, $y = ½$ and $z = -½$

From equation (1), $T = k\, m^0 l^{½} g^{-½}$

$$T = k\left[\frac{l}{g}\right]^{1/2} = k\sqrt{\frac{l}{g}}$$

Experimentally the value of k is determined to be 2π.

$$\therefore \quad T = 2\pi\sqrt{\frac{l}{g}}$$

1.9.2 Limitations of Dimensional Analysis

(i) The value of dimensionless constants cannot be determined by this method.

(ii) This method cannot be applied to equations involving exponential and trigonometric functions.

(iii) It cannot be applied to an equation involving more than three physical quantities.

(iv) It can check only whether a physical relation is dimensionally correct or not. It cannot tell whether the relation is absolutely correct or not. For example applying this technique $s = ut + \frac{1}{4}at^2$ is dimensionally correct whereas the correct relation is $s = ut + \frac{1}{2}at^2$.

Solved Problems

1.1 A laser signal is beamed towards a distant planet from the Earth and its reflection is received after seven minutes. If the distance between the planet and the Earth is 6.3×10^{10} m, calculate the velocity of the signal.

Data : $d = 6.3 \times 10^{10}$ m, t = 7 minutes = 7 × 60 = 420 s

Solution : *If d is the distance of the planet, then total distance travelled by the signal is 2d.*

$$\therefore \text{ velocity} = \frac{2d}{t} = \frac{2 \times 6.3 \times 10^{10}}{420} = 3 \times 10^{8} \text{ m s}^{-1}$$

1.2 A goldsmith put a ruby in a box weighing 1.2 kg. Find the total mass of the box and ruby applying principle of significant figures. The mass of the ruby is 5.42 g.

Data : *Mass of box = 1.2 kg*

Mass of ruby = 5.42 g = 5.42×10^{-3} kg = 0.00542 kg

Solution: *Total mass = mass of box + mass of ruby*

= 1.2 + 0.00542 = 1.20542 kg

After rounding off, total mass = 1.2 kg

1.3 Check whether the equation $\lambda = \frac{h}{mv}$ is dimensionally correct (λ - wavelength, h - Planck's constant, m - mass, v - velocity).

Solution: *Dimension of Planck's constant h is $[ML^2 T^{-1}]$*

Dimension of λ is $[L]$

Dimension of m is $[M]$

Dimension of v is $[LT^{-1}]$

Rewriting $\lambda = \frac{h}{mv}$ *using dimension*

$$[L] = \frac{[ML^2T^{-1}]}{[M][LT^{-1}]}$$

$$[L] = [L]$$

As the dimensions on both sides of the equation are same, the given equation is dimensionally correct.

1.4 Multiply 2.2 and 0.225. Give the answer correct to significant figures.

Solution : *$2.2 \times 0.225 = 0.495$*

Since the least number of significant figure in the given data is 2, the result should also have only two significant figures.

$$\therefore\ 2.2 \times 0.225 = 0.50$$

1.5 Convert 76 cm of mercury pressure into N m^{-2} using the method of dimensions.

Solution : *In cgs system, 76 cm of mercury pressure = $76 \times 13.6 \times 980$ dyne cm^{-2}*

Let this be P_1. Therefore $P_1 = 76 \times 13.6 \times 980$ dyne cm^{-2}

In cgs system, the dimension of pressure is $[M_1^a L_1^b T_1^c]$

Dimension of pressure is $[ML^{-1}T^{-2}]$. Comparing this with $[M_2^a L_2^b T_2^c]$ we have $a = 1$, $b = -1$ and $c = -2$.

$$\therefore \text{ Pressure in SI system} \quad P_2 = P_1 \left[\frac{M_1}{M_2}\right]^a \left[\frac{L_1}{L_2}\right]^b \left[\frac{T_1}{T_2}\right]^c$$

$$\text{ie } P_2 = 76 \times 13.6 \times 980 \left[\frac{10^{-3}\text{ kg}}{1\text{ kg}}\right]^1 \left[\frac{10^{-2}\text{ m}}{1\text{ m}}\right]^{-1} \left[\frac{1\text{s}}{1\text{s}}\right]^{-2}$$

$$= 76 \times 13.6 \times 980 \times 10^{-3} \times 10^2$$

$$= 101292.8 \text{ N m}^{-2}$$

$$P_2 = 1.01 \times 10^5 \text{ N m}^{-2}$$

2. Kinematics

Mechanics is one of the oldest branches of physics. It deals with the study of particles or bodies when they are at rest or in motion. Modern research and development in the spacecraft design, its automatic control, engine performance, electrical machines are highly dependent upon the basic principles of mechanics. *Mechanics* can be divided into *statics* and *dynamics*.

Statics is the study of objects at rest; this requires the idea of forces in equilibrium.

Dynamics is the study of moving objects. It comes from the Greek word *dynamis* which means power. *Dynamics* is further subdivided into *kinematics* and *kinetics*.

Kinematics is the study of the relationship between displacement, velocity, acceleration and time of a given motion, without considering the forces that cause the motion.

Kinetics deals with the relationship between the motion of bodies and forces acting on them.

We shall now discuss the various fundamental definitions in kinematics.

Particle

A particle is ideally just a piece or a quantity of matter, having practically no linear dimensions but only a position.

Rest and Motion

When a body does not change its position with respect to time, then it is said to be at rest.

Motion is the change of position of an object with respect to time. To study the motion of the object, one has to study the change in position (x,y,z coordinates) of the object with respect to the surroundings. It may be noted that the position of the object changes even due to the change in one, two or all the three coordinates of the position of the objects with respect to time. Thus motion can be classified into three types :

(i) Motion in one dimension

Motion of an object is said to be one dimensional, if only one of the three coordinates specifying the position of the object changes with respect to time. Example : An ant moving in a straight line, running athlete, etc.

(ii) Motion in two dimensions

In this type, the motion is represented by any two of the three coordinates. Example : a body moving in a plane.

(iii) Motion in three dimensions

Motion of a body is said to be three dimensional, if all the three coordinates of the position of the body change with respect to time.

Examples : motion of a flying bird, motion of a kite in the sky, motion of a molecule, etc.

2.1 Motion in one dimension (rectilinear motion)

The motion along a straight line is known as rectilinear motion. The important parameters required to study the motion along a straight line are position, displacement, velocity, and acceleration.

2.1.1 Position, displacement and distance travelled by the particle

The motion of a particle can be described if its position is known continuously with respect to time.

The total length of the path is the distance travelled by the particle and the shortest distance between the initial and final position of the particle is the displacement.

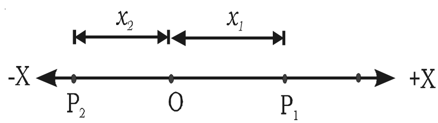


Fig 2.1 Distance and displacement

The distance travelled by a particle, however, is different from its displacement from the origin. For example, if the particle moves from a point O to position P_1 and then to position P_2, its displacement at the position P_2 is $-x_2$ from the origin but, the distance travelled by the particle is $x_1+x_1+x_2 = (2x_1+x_2)$ (Fig 2.1).

The distance travelled is a scalar quantity and the displacement is a vector quantity.

2.1.2 Speed and velocity

Speed

It is the distance travelled in unit time. It is a scalar quantity.

Velocity

The velocity of a particle is defined as the rate of change of displacement of the particle. It is also defined as the speed of the particle in a given direction. The velocity is a vector quantity. It has both magnitude and direction.

$$\text{Velocity} = \frac{\text{displacement}}{\text{time taken}}$$

Its unit is m s^{-1} and its dimensional formula is LT^{-1}.

Uniform velocity

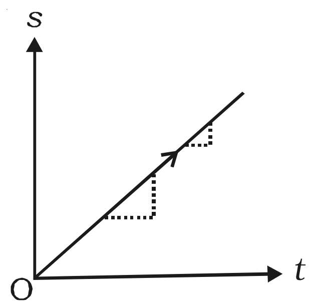


Fig. 2.2 Uniform velocity

A particle is said to move with uniform velocity if it moves along a fixed direction and covers equal displacements in equal intervals of time, however small these intervals of time may be.

In a displacement - time graph, (Fig. 2.2) the slope is constant at all the points, when the particle moves with uniform velocity.

Non uniform or variable velocity

The velocity is variable (non-uniform), if it covers unequal displacements in equal intervals of time or if the direction of motion changes or if both the rate of motion and the direction change.

Average velocity

Let s_1 be the displacement of a body in time t_1 and s_2 be its displacement in time t_2 (Fig. 2.3). The average velocity during the time interval $(t_2 - t_1)$ is defined as

$$v_{average} = \frac{\text{change in displacement}}{\text{change in time}}$$

$$= \frac{s_2 - s_1}{t_2 - t_1} = \frac{\Delta s}{\Delta t}$$

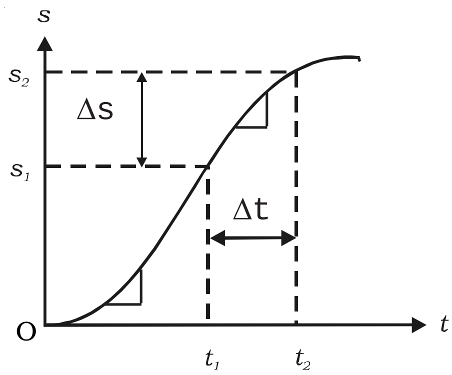


Fig. 2.3 Average velocity

From the graph, it is found that the slope of the curve varies.

Instantaneous velocity

It is the velocity at any given instant of time or at any given point of its path. The instantaneous velocity v is given by

$$v = \underset{\Delta t \to 0}{Lt} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$$

2.1.3 Acceleration

If the magnitude or the direction or both of the velocity changes with respect to time, the particle is said to be under acceleration.

Acceleration of a particle is defined as the rate of change of velocity. Acceleration is a vector quantity.

$$\text{Acceleration} = \frac{\text{change in velocity}}{\text{time taken}}$$

If u is the initial velocity and v, the final velocity of the particle after a time t, then the acceleration,

$$a = \frac{v - u}{t}$$

Its unit is m s^{-2} and its dimensional formula is LT^{-2}.

The instantaneous acceleration is, $a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}$

Uniform acceleration

If the velocity changes by an equal amount in equal intervals of time, however small these intervals of time may be, the acceleration is said to be uniform.

Retardation or deceleration

If the velocity decreases with time, the acceleration is negative. The negative acceleration is called retardation or deceleration.

Uniform motion

A particle is in uniform motion when it moves with constant velocity (i.e) zero acceleration.

2.1.4 Graphical representations

The graphs provide a convenient method to present pictorially, the basic informations about a variety of events. Line graphs are used to show the relation of one quantity say displacement or velocity with another quantity such as time.

If the displacement, velocity and acceleration of a particle are plotted with respect to time, they are known as,

(i) displacement – time graph (s - t graph)

(ii) velocity – time graph (v - t graph)

(iii) acceleration – time graph (a - t graph)

Displacement – time graph

When the displacement of the particle is plotted as a function of time, it is displacement - time graph.

As $v = \frac{ds}{dt}$, the slope of the s - t graph at any instant gives the velocity of the particle at that instant. In Fig. 2.4 the particle at time t_1, has a positive velocity, at time t_2, has zero velocity and at time t_3, has negative velocity.

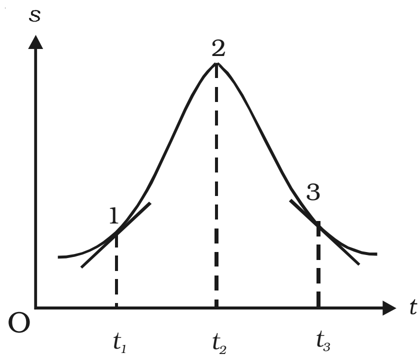


Fig. 2.4 Displacement - time graph

Velocity – time graph

When the velocity of the particle is plotted as a function of time, it is velocity-time graph.

As $a = \frac{dv}{dt}$, the slope of the $v - t$ curve at any instant gives the acceleration of the particle (Fig. 2.5).

But, $v = \frac{ds}{dt}$ or $ds = v.dt$

If the displacements are s_1 and s_2 in times t_1 and t_2, then

$$\int_{s_1}^{s_2} ds = \int_{t_1}^{t_2} v\,dt$$

$$s_2 - s_1 = \int_{t_1}^{t_2} v\,dt = \text{area ABCD}$$

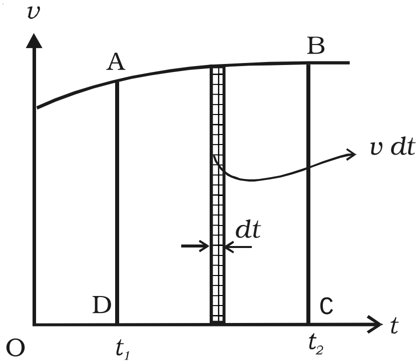


Fig. 2.5 Velocity - time graph

The area under the $v - t$ curve, between the given intervals of time, gives the change in displacement or the distance travelled by the particle during the same interval.

Acceleration – time graph

When the acceleration is plotted as a function of time, it is acceleration - time graph (Fig. 2.6).

$a = \frac{dv}{dt}$ (or) $dv = a\,dt$

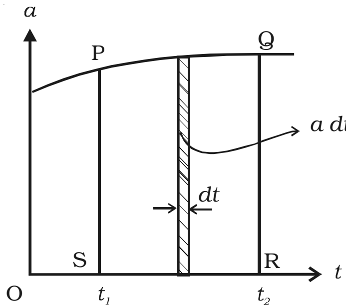


Fig. 2.6 Acceleration – time graph

If the velocities are v_1 and v_2 at times t_1 and t_2 respectively, then

$$\int_{v_1}^{v_2} dv = \int_{t_1}^{t_2} a\,dt \quad \text{(or)} \quad v_2 - v_1 = \int_{t_1}^{t_2} a.dt = \text{area PQRS}$$

The area under the $a - t$ curve, between the given intervals of time, gives the change in velocity of the particle during the same interval. If the graph is parallel to the time axis, the body moves with constant acceleration.

2.1.5 Equations of motion

For uniformly accelerated motion, some simple equations that relate displacement s, time t, initial velocity u, final velocity v and acceleration a are obtained.

(i) As acceleration of the body at any instant is given by the first derivative of the velocity with respect to time,

$$a = \frac{dv}{dt} \text{ (or) } dv = a.dt$$

If the velocity of the body changes from u to v in time t then from the above equation,

$$\int_u^v dv = \int_0^t a\ dt = a\int_0^t dt \Rightarrow [v]_u^v = a[t]_0^t$$

$$\therefore \quad v - u = at \quad \text{(or)} \quad v = u + at \qquad ...(1)$$

(ii) The velocity of the body is given by the first derivative of the displacement with respect to time.

(i.e) $v = \dfrac{ds}{dt}$ (or) $ds = v\,dt$

Since $v = u + at$, $ds = (u + at)\ dt$

The distance s covered in time t is,

$$\int_0^s ds = \int_0^t u\,dt + \int_0^t at\,dt \quad \text{(or)} \quad s = ut + \frac{1}{2}at^2 \qquad ...(2)$$

(iii) The acceleration is given by the first derivative of velocity with respect to time. (i.e)

$$a = \frac{dv}{dt} = \frac{dv}{ds} \cdot \frac{ds}{dt} = \frac{dv}{ds} \cdot v \left[\because v = \frac{ds}{dt}\right] \quad \text{(or)} \quad ds = \frac{1}{a}\ v\,dv$$

Therefore,

$$\int_0^s ds = \int_u^v \frac{v\ dv}{a} \quad \text{(i.e)} \quad s = \frac{1}{a}\left[\frac{v^2}{2} - \frac{u^2}{2}\right]$$

$$s = \frac{1}{2a}\left(v^2 - u^2\right) \text{ (or) } 2as = (v^2 - u^2)$$

$$\therefore \quad v^2 = u^2 + 2\ as \qquad ...(3)$$

The equations (1), (2) and (3) are called equations of motion.

Expression for the distance travelled in n^{th} second

Let a body move with an initial velocity u and travel along a straight line with uniform acceleration a.

Distance travelled in the n^{th} second of motion is,

s_n = *distance travelled during first n seconds – distance travelled during (n –1) seconds*

Distance travelled during n seconds

$$D_n = un + \frac{1}{2}an^2$$

Distance travelled during $(n-1)$ seconds

$$D_{(n-1)} = u(n-1) + \frac{1}{2}a(n-1)^2$$

$\therefore$ the distance travelled in the n^{th} second = $D_n - D_{(n-1)}$

$$\text{(i.e)}\quad s_n = \left(un + \frac{1}{2}an^2\right) - \left[u(n-1) + \frac{1}{2}a(n-1)^2\right]$$

$$s_n = u + a\left(n - \frac{1}{2}\right)$$

$$s_n = u + \frac{1}{2}a(2n-1)$$

Special Cases

Case (i) : For downward motion

For a particle moving downwards, $a = g$, since the particle moves in the direction of gravity.

Case (ii) : For a freely falling body

For a freely falling body, $a = g$ and $u = 0$, since it starts from rest.

Case (iii) : For upward motion

For a particle moving upwards, $a = -g$, since the particle moves against the gravity.

2.2 Scalar and vector quantities

A study of motion will involve the introduction of a variety of quantities, which are used to describe the physical world. Examples of such quantities are distance, displacement, speed, velocity, acceleration, mass, momentum, energy, work, power etc. All these quantities can be divided into two categories – *scalars* and *vectors*.

The scalar quantities have magnitude only. It is denoted by a number and unit. Examples : length, mass, time, speed, work, energy, temperature etc. Scalars of the same kind can be added, subtracted, multiplied or divided by ordinary laws.

The vector quantities have both magnitude and direction. Examples: displacement, velocity, acceleration, force, weight, momentum, etc.

2.2.1 Representation of a vector

Vector quantities are often represented by a scaled vector diagrams. Vector diagrams represent a vector by the use of an arrow drawn to scale in a specific direction. An example of a scaled vector diagram is shown in Fig 2.7.

From the figure, it is clear that

(i) The scale is listed.

(ii) A line with an arrow is drawn in a specified direction.

(iii) The magnitude and direction of the vector are clearly labelled. In the above case, the diagram shows that the magnitude is 4 N and direction is 30° to x-axis. The length of the line gives the magnitude and arrow head gives the direction. In notation, the vector is denoted in bold face letter such as ***A*** or with an arrow above the letter as $\vec{A}$, read as vector *A* or *A* vector while its magnitude is denoted by *A* or by $|\vec{A}|$.

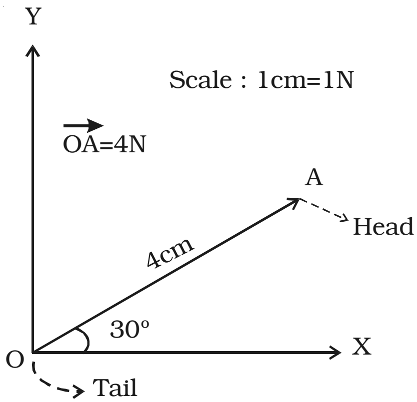


Fig 2.7 Representation of a vector

2.2.2 Different types of vectors

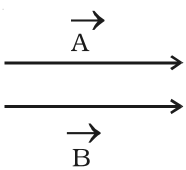


Fig. 2.8 Equal vectors

(i) Equal vectors

Two vectors are said to be equal if they have the same magnitude and same direction, wherever be their initial positions. In Fig. 2.8 the vectors $\vec{A}$ and $\vec{B}$ have the same magnitude and direction. Therefore $\vec{A}$ and $\vec{B}$ are equal vectors.

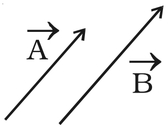


Fig. 2.9
Like vectors

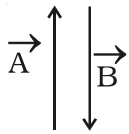


Fig. 2.10
Opposite vectors

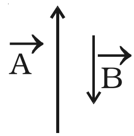


Fig. 2.11
Unlike Vectors

(ii) Like vectors

Two vectors are said to be like vectors, if they have same direction but different magnitudes as shown in Fig. 2.9.

(iii) Opposite vectors

The vectors of same magnitude but opposite in direction, are called opposite vectors (Fig. 2.10).

(iv) Unlike vectors

The vectors of different magnitude acting in opposite directions are called unlike vectors. In Fig. 2.11 the vectors $\vec{A}$ and $\vec{B}$ are unlike vectors.

(v) Unit vector

A vector having unit magnitude is called a unit vector. It is also defined as a vector divided by its own magnitude. A unit vector in the direction of a vector $\vec{A}$ is written as $\hat{A}$ and is read as 'A cap' or 'A caret' or 'A hat'. Therefore,

$$\hat{A} = \frac{\vec{A}}{|\vec{A}|} \quad \text{(or)} \quad \vec{A} = \hat{A}\,|\vec{A}|$$

Thus, a vector can be written as the product of its magnitude and unit vector along its direction.

Orthogonal unit vectors

There are three most common unit vectors in the positive directions of X,Y and Z axes of Cartesian coordinate system, denoted by i, j and k respectively. Since they are along the mutually perpendicular directions, they are called orthogonal unit vectors.

(vi) Null vector or zero vector

A vector whose magnitude is zero, is called a null vector or zero vector. It is represented by $\vec{0}$ and its starting and end points are the same. The direction of null vector is not known.

(vii) Proper vector

All the non-zero vectors are called proper vectors.

(viii) Co-initial vectors

Vectors having the same starting point are called co-initial vectors. In Fig. 2.12, $\vec{A}$ and $\vec{B}$ start from the same origin O. Hence, they are called as co-initial vectors.

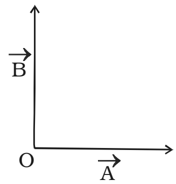


Fig 2.12
Co-initial vectors

(ix) Coplanar vectors

Vectors lying in the same plane are called coplanar vectors and the plane in which the vectors lie are called plane of vectors.

2.2.3 Addition of vectors

As vectors have both magnitude and direction they cannot be added by the method of ordinary algebra.

Vectors can be added graphically or geometrically. We shall now discuss the addition of two vectors graphically using head to tail method.

Consider two vectors $\vec{P}$ and $\vec{Q}$ which are acting along the same line. To add these two vectors, join the tail of $\vec{Q}$ with the head of $\vec{P}$ (Fig. 2.13).

The resultant of $\vec{P}$ and $\vec{Q}$ is $\vec{R} = \vec{P} + \vec{Q}$. The length of the line AD gives the magnitude of $\vec{R}$. $\vec{R}$ acts in the same direction as that of $\vec{P}$ and $\vec{Q}$.

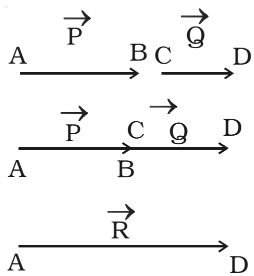


Fig. 2.13
Addition of vectors

In order to find the sum of two vectors, which are inclined to each other, triangle law of vectors or parallelogram law of vectors, can be used.

(i) Triangle law of vectors

If two vectors are represented in magnitude and direction by the two adjacent sides of a triangle taken in order, then their resultant is the closing side of the triangle taken in the reverse order.

To find the resultant of two vectors $\vec{P}$ and $\vec{Q}$ which are acting at an angle θ, the following procedure is adopted.

First draw $\overline{OA} = \vec{P}$ (Fig. 2.14) Then starting from the arrow head of $\vec{P}$, draw the vector $\overrightarrow{AB} = \vec{Q}$. Finally, draw a vector $\overrightarrow{OB} = \vec{R}$ from the tail of vector $\vec{P}$ to the head of vector $\vec{Q}$. Vector $\overrightarrow{OB} = \vec{R}$ is the sum of the vectors $\vec{P}$ and $\vec{Q}$. Thus $\vec{R} = \vec{P} + \vec{Q}$.

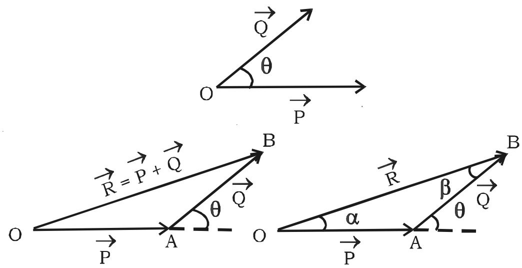


Fig. 2.14 Triangle law of vectors

The magnitude of $\vec{P} + \vec{Q}$ is determined by measuring the length of $\vec{R}$ and direction by measuring the angle between $\vec{P}$ and $\vec{R}$.

The magnitude and direction of $\vec{R}$, can be obtained by using the *sine law* and *cosine law* of triangles. Let α be the angle made by the resultant $\vec{R}$ with $\vec{P}$. The magnitude of $\vec{R}$ is,

$$R^2 = P^2 + Q^2 - 2PQ \cos (180^o - \theta)$$

$$R = \sqrt{P^2 + Q^2 + 2PQ \cos \theta}$$

The direction of R can be obtained by,

$$\frac{P}{\sin \beta} = \frac{Q}{\sin \alpha} = \frac{R}{\sin (180^o - \theta)}$$

(ii) Parallelogram law of vectors

If two vectors acting at a point are represented in magnitude and direction by the two adjacent sides of a parallelogram, then their resultant is represented in magnitude and direction by the diagonal passing through the common tail of the two vectors.

Let us consider two vectors $\vec{P}$ and $\vec{Q}$ which are inclined to each other at an angle θ as shown in Fig. 2.15. Let the vectors $\vec{P}$ and $\vec{Q}$ be represented in magnitude and direction by the two sides OA and OB of a parallelogram $OACB$. The diagonal OC passing through the common tail O, gives the magnitude and direction of the resultant $\vec{R}$.

CD is drawn perpendicular to the extended OA, from C. Let $\underline{|COD}$ made by $\vec{R}$ with $\vec{P}$ be α.

From right angled triangle OCD,

$$OC^2 = OD^2 + CD^2$$

$$= (OA + AD)^2 + CD^2$$

$$= OA^2 + AD^2 + 2.OA.AD + CD^2 \quad ...(1)$$

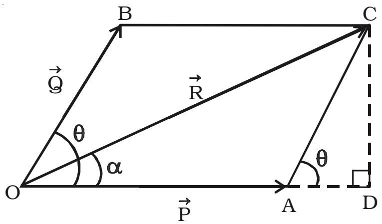


Fig 2.15 Parallelogram law of vectors

In Fig. 2.15 $\underline{|BOA} = \theta = \underline{|CAD}$

From right angled Δ CAD,

$$AC^2 = AD^2 + CD^2 \quad ...(2)$$

Substituting (2) in (1)

$$OC^2 = OA^2 + AC^2 + 2OA.AD \quad ...(3)$$

From ΔACD,

$$CD = AC \sin \theta \quad ...(4)$$

$$AD = AC \cos \theta \quad ...(5)$$

Substituting (5) in (3) $OC^2 = OA^2 + AC^2 + 2\ OA.AC \cos \theta$

Substituting $OC = R$, $OA = P$,

$OB = AC = Q$ in the above equation

$$R^2 = P^2 + Q^2 + 2PQ \cos \theta$$

(or) $$R = \sqrt{P^2 + Q^2 + 2PQ \cos \theta} \quad ...(6)$$

Equation (6) gives the magnitude of the resultant. From $\Delta\ OCD$,

$$\tan \alpha = \frac{CD}{OD} = \frac{CD}{OA + AD}$$

Substituting (4) and (5) in the above equation,

$$\tan \alpha = \frac{AC \sin \theta}{OA + AC \cos \theta} = \frac{Q \sin \theta}{P + Q \cos \theta}$$

(or) $$\alpha = \tan^{-1}\left[\frac{Q \sin \theta}{P + Q \cos \theta}\right] \quad ...(7)$$

Equation (7) gives the direction of the resultant.

Special Cases

(i) When two vectors act in the same direction

In this case, the angle between the two vectors $\theta = 0°$, $\cos 0° = 1$, $\sin 0° = 0$

From (6) $$R = \sqrt{P^2 + Q^2 + 2PQ} = (P + Q)$$

From (7) $$\alpha = \tan^{-1}\left[\frac{Q \sin 0^o}{P + Q \cos 0^o}\right]$$

(i.e) $\alpha = 0$

Thus, the resultant vector acts in the same direction as the individual vectors and is equal to the sum of the magnitude of the two vectors.

(ii) When two vectors act in the opposite direction

In this case, the angle between the two vectors $\theta = 180°$, $\cos 180° = -1$, $\sin 180° = 0$.

From (6) $$R = \sqrt{P^2 + Q^2 - 2PQ} = (P - Q)$$

From (7) $$\alpha = \tan^{-1}\left[\frac{0}{P - Q}\right] = \tan^{-1}(0) = 0$$

Thus, the resultant vector has a magnitude equal to the difference in magnitude of the two vectors and acts in the direction of the bigger of the two vectors

(iii) When two vectors are at right angles to each other

In this case, $\theta = 90°$, $\cos 90° = 0$, $\sin 90° = 1$

From (6) $$R = \sqrt{P^2 + Q^2}$$

From (7) $$\alpha = \tan^{-1}\left(\frac{Q}{P}\right)$$

The resultant $\vec{R}$ vector acts at an angle α with vector $\vec{P}$.

2.2.4 Subtraction of vectors

The subtraction of a vector from another is equivalent to the addition of one vector to the negative of the other.

For example $\vec{Q} - \vec{P} = \vec{Q} + (-\vec{P})$.

Thus to subtract $\vec{P}$ from $\vec{Q}$, one has to add $-\vec{P}$ with $\vec{Q}$ (Fig 2.16a). Therefore, to subtract $\vec{P}$ from $\vec{Q}$, reversed $\vec{P}$ is added to the

$\vec{Q}$. For this, first draw $\overrightarrow{AB} = \vec{Q}$ and then starting from the arrow head of $\vec{Q}$, draw $\overrightarrow{BC} = (-\vec{P})$ and finally join the head of $-\vec{P}$. Vector $\vec{R}$ is the sum of $\vec{Q}$ and $-\vec{P}$. (i.e) difference $\vec{Q} - \vec{P}$.

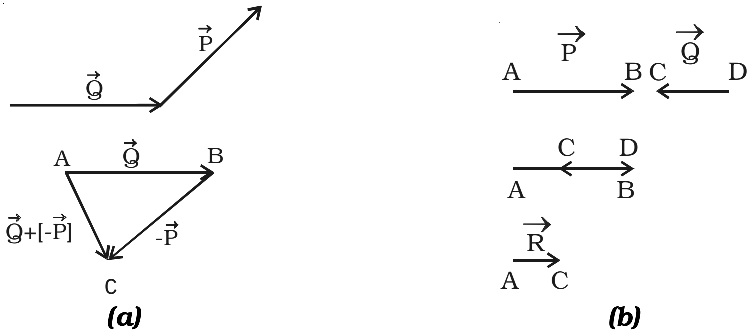


Fig 2.16 Subtraction of vectors

The resultant of two vectors which are antiparallel to each other is obtained by subtracting the smaller vector from the bigger vector as shown in Fig 2.16b. The direction of the resultant vector is in the direction of the bigger vector.

2.2.5 Product of a vector and a scalar

Multiplication of a scalar and a vector gives a vector quantity which acts along the direction of the vector.

Examples

(i) If $\vec{a}$ is the acceleration produced by a particle of mass m under the influence of the force, then $\vec{F} = m\vec{a}$

(ii) momentum = mass × velocity (i.e) $\vec{P} = m\vec{v}$.

2.2.6 Resolution of vectors and rectangular components

A vector directed at an angle with the co-ordinate axis, can be resolved into its components along the axes. This process of splitting a vector into its components is known as resolution of a vector.

Consider a vector $\vec{R} = \overrightarrow{OA}$ making an angle θ with X - axis. The vector R can be resolved into two components along X - axis and Y-axis respectively. Draw two perpendiculars from A to X and Y axes respectively. The intercepts on these axes are called the scalar components R_x and R_y.

Then, OP is R_x, which is the magnitude of x component of $\vec{R}$ and OQ is R_y, which is the magnitude of y component of $\vec{R}$

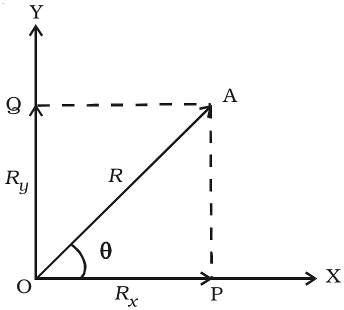


Fig. 2.17 Rectangular components of a vector

From Δ OPA,

$$\cos\theta = \frac{OP}{OA} = \frac{R_x}{R} \quad \text{(or)} \quad R_x = R\cos\theta$$

$$\sin\theta = \frac{OQ}{OA} = \frac{R_y}{R} \quad \text{(or)} \quad R_y = R\sin\theta$$

$$\text{and} \quad R^2 = R_x^2 + R_y^2$$

Also, $\vec{R}$ can be expressed as $\vec{R} = R_x\vec{i} + R_y\vec{j}$ where i and j are unit vectors.

In terms of R_x and R_y, θ can be expressed as $\theta = \tan^{-1}\left[\frac{R_y}{R_x}\right]$

2.2.7 Multiplication of two vectors

Multiplication of a vector by another vector does not follow the laws of ordinary algebra. There are two types of vector multiplication (i) Scalar product and (ii) Vector product.

(i) Scalar product or Dot product of two vectors

If the product of two vectors is a scalar, then it is called scalar product. If $\vec{A}$ and $\vec{B}$ are two vectors, then their scalar product is written as $\vec{A}.\vec{B}$ and read as $\vec{A}$ dot $\vec{B}$. Hence scalar product is also called dot product. This is also referred as inner or direct product.

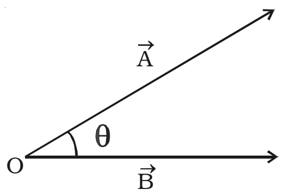


Fig 2.18 Scalar product of two vectors

The scalar product of two vectors is a scalar, which is equal to the product of magnitudes of the two vectors and the cosine of the angle between them. The scalar product of two vectors $\vec{A}$ and $\vec{B}$ may be expressed as $\vec{A} \cdot \vec{B} = |\vec{A}|\ |\vec{B}| \cos\theta$ where $|\vec{A}|$ and $|\vec{B}|$ are the magnitudes of $\vec{A}$ and $\vec{B}$ respectively and θ is the angle between $\vec{A}$ and $\vec{B}$ as shown in Fig 2.18.

(ii) Vector product or Cross product of two vectors

If the product of two vectors is a vector, then it is called vector product. If $\vec{A}$ and $\vec{B}$ are two vectors then their vector product is written as $\vec{A} \times \vec{B}$ and read as $\vec{A}$ cross $\vec{B}$. This is also referred as outer product.

The vector product or cross product of two vectors is a vector whose magnitude is equal to the product of their magnitudes and the sine of the smaller angle between them and the direction is perpendicular to a plane containing the two vectors.

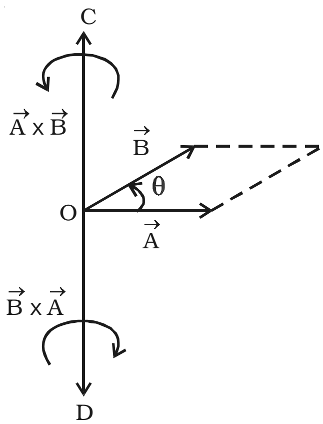


Fig 2.19 Vector product of two vectors

If θ is the smaller angle through which $\vec{A}$ should be rotated to reach $\vec{B}$, then the cross product of $\vec{A}$ and $\vec{B}$ (Fig. 2.19) is expressed as,

$$\vec{A} \times \vec{B} = |\vec{A}|\ |\vec{B}|\ \sin\theta\ \hat{n} = \vec{C}$$

where $|\vec{A}|$ and $|\vec{B}|$ are the magnitudes of $\vec{A}$ and $\vec{B}$ respectively. $\vec{C}$ is perpendicular to the plane containing $\vec{A}$ and $\vec{B}$. The direction of $\vec{C}$ is along the direction in which the tip of a screw moves when it is rotated from $\vec{A}$ to $\vec{B}$. Hence $\vec{C}$ acts along OC. By the same argument, $\vec{B} \times \vec{A}$ acts along OD.

2.3 Projectile motion

A body thrown with some initial velocity and then allowed to move under the action of gravity alone, is known as a projectile.

If we observe the path of the projectile, we find that the projectile moves in a path, which can be considered as a part of parabola. Such a motion is known as *projectile motion.*

A few examples of projectiles are (i) a bomb thrown from an aeroplane (ii) a javelin or a shot-put thrown by an athlete (iii) motion of a ball hit by a cricket bat etc.

The different types of projectiles are shown in Fig. 2.20. A body can be projected in two ways:

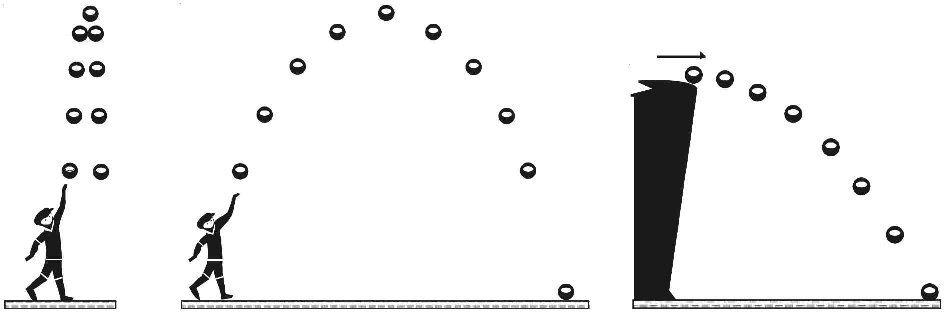


Fig 2.20 Different types of projectiles

(i) It can be projected horizontally from a certain height.

(ii) It can be thrown from the ground in a direction inclined to it.

The projectiles undergo a vertical motion as well as horizontal motion. The two components of the projectile motion are (i) vertical component and (ii) horizontal component. These two perpendicular components of motion are independent of each other.

A body projected with an initial velocity making an angle with the horizontal direction possess uniform horizontal velocity and variable vertical velocity, due to force of gravity. The object therefore has horizontal and vertical motions simultaneously. The resultant motion would be the vector sum of these two motions and the path following would be curvilinear.

The above discussion can be summarised as in the Table 2.1

Table 2.1 Two independent motions of a projectile

Motion	**Forces**	**Velocity**	**Acceleration**
Horizontal	No force acts	Constant	Zero
Vertical	The force of gravity acts downwards	Changes ($\sim$10 m s^{-1})	Downwards ($\sim$10 m s^{-2})

In the study of projectile motion, it is assumed that the air resistance is negligible and the acceleration due to gravity remains constant.

Angle of projection

The angle between the initial direction of projection and the horizontal direction through the point of projection is called the angle of projection.

Velocity of projection

The velocity with which the body is projected is known as velocity of projection.

Range

Range of a projectile is the horizontal distance between the point of projection and the point where the projectile hits the ground.

Trajectory

The path described by the projectile is called the trajectory.

Time of flight

Time of flight is the total time taken by the projectile from the instant of projection till it strikes the ground.

2.3.1 Motion of a projectile thrown horizontally

Let us consider an object thrown horizontally with a velocity u from a point A, which is at a height h from the horizontal plane OX (Fig 2.21). The object acquires the following motions simultaneously :

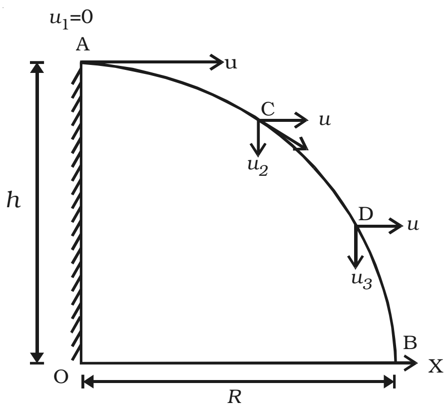


Fig 2.21 Projectile projected horizontally from the top of a tower

(i) Uniform velocity with which it is projected in the horizontal direction OX

(ii) Vertical velocity, which is non-uniform due to acceleration due to gravity.

The two velocities are independent of each other. The horizontal velocity of the object shall remain constant as no acceleration is acting in the horizontal direction. The velocity in the vertical direction shall go on changing because of acceleration due to gravity.

Path of a projectile

Let the time taken by the object to reach C from A = t

Vertical distance travelled by the object in time $t = s = y$

From equation of motion, $s = u_1 t + \frac{1}{2} at^2$ ...(1)

Substituting the known values in equation (1),

$$y = (0)\, t + \frac{1}{2} gt^2 = \frac{1}{2} gt^2 \qquad ...(2)$$

At A, the initial velocity in the horizontal direction is u.

Horizontal distance travelled by the object in time t is x.

$\therefore \quad x$ = horizontal velocity × time $= u\,t$ (or) $t = \frac{x}{u}$ ...(3)

Substituting t in equation (2), $y = \frac{1}{2} g \left(\frac{x}{u}\right)^2 = \frac{1}{2} g \frac{x^2}{u^2}$ ...(4)

(or) $y = kx^2$

where $k = \frac{g}{2u^2}$ is a constant.

The above equation is the equation of a parabola. Thus *the path taken by the projectile is a parabola.*

Resultant velocity at C

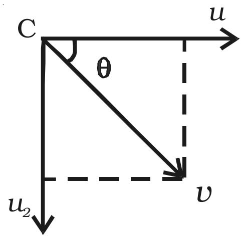


Fig 2.22
Resultant velocity at any point

At an instant of time t, let the body be at C.

At A, initial vertical velocity $(u_1) = 0$

At C, the horizontal velocity $(u_x) = u$

At C, the vertical velocity $= u_2$

From equation of motion, $u_2 = u_1 + g\,t$

Substituting all the known values, $u_2 = 0 + g\,t$ ...(5)

The resultant velocity at C is $v = \sqrt{u_x^2 + u_2^2} = \sqrt{u^2 + g^2 t^2}$ ...(6)

The direction of v is given by $\tan\theta = \frac{u_2}{u_x} = \frac{gt}{u}$ ...(7)

where θ is the angle made by v with X axis.

Time of flight and range

The distance $OB = R$, is called as range of the projectile.

Range = horizontal velocity × time taken to reach the ground

$$R = u\, t_f \quad ...(8)$$

where t_f is the time of flight

At A, initial vertical velocity $(u_1) = 0$

The vertical distance travelled by the object in time $t_f = s_y = h$

From the equations of motion $$S_y = u_1 t_f + \frac{1}{2} g\, t_f^2 \quad ...(9)$$

Substituting the known values in equation (9),

$$h = (0)\, t_f + \frac{1}{2} g\, t_f^2 \quad \text{(or)} \quad t_f = \sqrt{\frac{2h}{g}} \quad ...(10)$$

Substituting t_f in equation (8), Range $$R = u\sqrt{\frac{2h}{g}} \quad ...(11)$$

2.3.2 Motion of a projectile projected at an angle with the horizontal (oblique projection)

Consider a body projected from a point O on the surface of the Earth with an initial velocity u at an angle θ with the horizontal as shown in Fig. 2.23. The velocity u can be resolved into two components

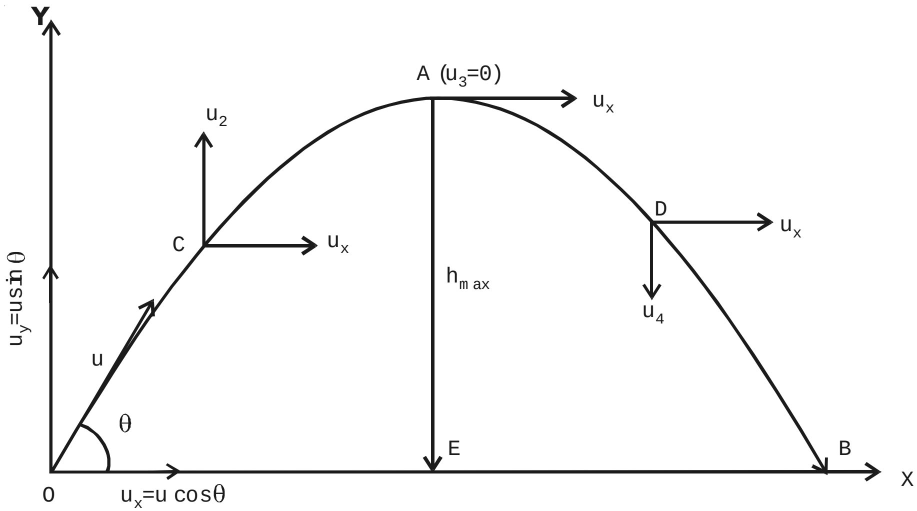


Fig 2.23 Motion of a projectile projected at an angle with horizontal

(i) $u_x = u \cos \theta$, along the horizontal direction OX and

(ii) $u_y = u \sin \theta$, along the vertical direction OY

The horizontal velocity u_x of the object shall remain constant as no acceleration is acting in the horizontal direction. But the vertical component u_y of the object continuously decreases due to the effect of the gravity and it becomes zero when the body is at the highest point of its path. After this, the vertical component u_y is directed downwards and increases with time till the body strikes the ground at *B*.

Path of the projectile

Let t_1 be the time taken by the projectile to reach the point C from the instant of projection.

Horizontal distance travelled by the projectile in time t_1 is,

x = horizontal velocity × time

$$x = u \cos \theta \times t_1 \quad \text{(or)} \quad t_1 = \frac{x}{u \cos \theta} \qquad \text{...(1)}$$

Let the vertical distance travelled by the projectile in time $t_1 = s = y$

At O, initial vertical velocity $u_1 = u \sin \theta$

From the equation of motion $s = u_1 t_1 - \frac{1}{2} g t_1^2$

Substituting the known values,

$$y = (u \sin \theta)\, t_1 - \frac{1}{2} g t_1^2 \qquad \text{...(2)}$$

Substituting equation (1) in equation (2),

$$y = (u \sin \theta) \left(\frac{x}{u \cos \theta} \right) - \frac{1}{2}(g) \left(\frac{x}{u \cos \theta} \right)^2$$

$$y = x \tan \theta - \frac{g x^2}{2u^2 \cos^2 \theta} \qquad \text{...(3)}$$

The above equation is of the form $y = Ax + Bx^2$ and represents a parabola. Thus the path of a projectile is a parabola.

Resultant velocity of the projectile at any instant t_1

At C, the velocity along the horizontal direction is $u_x = u \cos \theta$ and the velocity along the vertical direction is $u_y = u_2$.

From the equation of motion,

$$u_2 = u_1 - gt_1$$

$$u_2 = u \sin\theta - gt_1$$

∴ The resultant velocity at

C is $v = \sqrt{u_x^2 + u_2^2}$

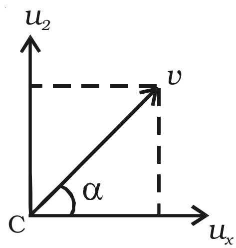


Fig 2.24 Resultant velocity of the projectile at any instant

$$v = \sqrt{(u\cos\theta)^2 + (u\sin\theta - gt_1)^2}$$

$$= \sqrt{u^2 + g^2t_1^2 - 2ut_1 g\sin\theta}$$

The direction of v is given by

$$\tan\alpha = \frac{u_2}{u_x} = \frac{u\sin\theta - gt_1}{u\cos\theta} \quad \text{(or)} \quad \alpha = \tan^{-1}\left[\frac{u\sin\theta - gt_1}{u\cos\theta}\right]$$

where α is the angle made by v with the horizontal line.

Maximum height reached by the projectile

The maximum vertical displacement produced by the projectile is known as the maximum height reached by the projectile. In Fig 2.23, EA is the maximum height attained by the projectile. It is represented as h_{max}.

At O, the initial vertical velocity $(u_1) = u \sin\theta$

At *A*, the final vertical velocity $(u_3) = 0$

The vertical distance travelled by the object = $s_y = h_{max}$

From equation of motion, $u_3^2 = u_1^2 - 2gs_y$

Substituting the known values, $(0)^2 = (u\sin\theta)^2 - 2gh_{max}$

$$2gh_{max} = u^2\sin^2\theta \quad \text{(or)} \quad h_{max} = \frac{u^2\sin^2\theta}{2g} \qquad ...(4)$$

Time taken to attain maximum height

Let *t* be the time taken by the projectile to attain its maximum height.

From equation of motion $u_3 = u_1 - g\,t$

Substituting the known values $0 = u \sin\theta - g t$

$$g\, t = u \sin\theta$$

$$t = \frac{u \sin\theta}{g} \qquad \text{...(5)}$$

Time of flight

Let t_f be the time of flight (i.e) *the time taken by the projectile to reach B from O through* A. When the body returns to the ground, the net vertical displacement made by the projectile

$$s_y = h_{max} - h_{max} = 0$$

From the equation of motion $s_y = u_1\, t_f - \frac{1}{2} g\, t_f^2$

Substituting the known values $0 = (\, u \sin\theta\,)\, t_f - \frac{1}{2} g\, t_f^2$

$$\frac{1}{2} g\, t_f^2 = (u \sin\theta)\, t_f \quad \text{(or)} \quad t_f = \frac{2u \sin\theta}{g} \qquad \text{...(6)}$$

From equations (5) and (6) $t_f = 2t$ ...(7)

(i.e) *the time of flight is twice the time taken to attain the maximum height.*

Horizontal range

The horizontal distance OB is called the range of the projectile.

Horizontal range = horizontal velocity × time of flight

(i.e) $R = u \cos\theta \times t_f$

Substituting the value of t_f, $R = (u \cos\theta) \frac{2u \sin\theta}{g}$

$$R = \frac{u^2\,(2 \sin\theta \cos\theta)}{g}$$

$$\therefore \quad R = \frac{u^2 \sin 2\theta}{g} \qquad \text{...(8)}$$

Maximum Range

From (8), it is seen that for the given velocity of projection, the horizontal range depends on the angle of projection only. The range is maximum only if the value of sin 2θ is maximum.

For maximum range R_{max} $\sin 2\theta = 1$

(i.e) $\theta = 45°$

Therefore *the range is maximum when the angle of projection is 45°.*

$$R_{max} = \frac{u^2 \times 1}{g} \Rightarrow R_{max} = \frac{u^2}{g} \quad ...(9)$$

2.4 Newton's laws of motion

Various philosophers studied the basic ideas of cause of motion. According to Aristotle, a constant external force must be applied continuously to an object in order to keep it moving with uniform velocity. Later this idea was discarded and Galileo gave another idea on the basis of the experiments on an inclined plane. According to him, no force is required to keep an object moving with constant velocity. It is the presence of frictional force that tends to stop moving object, the smaller the frictional force between the object and the surface on which it is moving, the larger the distance it will travel before coming to rest. After Galileo, it was Newton who made a systematic study of motion and extended the ideas of Galileo.

Newton formulated the laws concerning the motion of the object. There are three laws of motion. A deep analysis of these laws lead us to the conclusion that these laws completely define the force. The first law gives the fundamental definition of force; the second law gives the quantitative and dimensional definition of force while the third law explains the nature of the force.

2.4.1 Newton's first law of motion

It states that *every body continues in its state of rest or of uniform motion along a straight line unless it is compelled by an external force to change that state.*

This law is based on Galileo's law of inertia. Newton's first law of motion deals with the basic property of matter called inertia and the definition of force.

Inertia is that property of a body by virtue of which the body is unable to change its state by itself in the absence of external force.

The inertia is of three types

(i) Inertia of rest

(ii) Inertia of motion

(iii) Inertia of direction.

(i) Inertia of rest

It is the inability of the body to change its state of rest by itself.

Examples

(i) A person standing in a bus falls backward when the bus suddenly starts moving. This is because, the person who is initially at rest continues to be at rest even after the bus has started moving.

(ii) A book lying on the table will remain at rest, until it is moved by some external agencies.

(iii) When a carpet is beaten by a stick, the dust particles fall off vertically downwards once they are released and do not move along the carpet and fall off.

(ii) Inertia of motion

Inertia of motion is the inability of the body to change its state of motion by itself.

Examples

(a) When a passenger gets down from a moving bus, he falls down in the direction of the motion of the bus.

(b) A passenger sitting in a moving car falls forward, when the car stops suddenly.

(c) An athlete running in a race will continue to run even after reaching the finishing point.

(iii) Inertia of direction

It is the inability of the body to change its direction of motion by itself.

Examples

When a bus moving along a straight line takes a turn to the right, the passengers are thrown towards left. This is due to inertia which makes the passengers travel along the same straight line, even though the bus has turned towards the right.

This inability of a body to change by itself its state of rest or of uniform motion along a straight line or direction, is known as inertia. The inertia of a body is directly proportional to the mass of the body.

From the first law, we infer that to change the state of rest or uniform motion, an external agency called, the force is required.

Force is defined as that which when acting on a body changes or tends to change the state of rest or of uniform motion of the body along a straight line.

A force is a push or pull upon an object, resulting the change of state of a body. Whenever there is an interaction between two objects, there is a force acting on each other. When the interaction ceases, the two objects no longer experience a force. Forces exist only as a result of an interaction.

There are two broad categories of forces between the objects, contact forces and non–contact forces resulting from action at a distance.

Contact forces are forces in which the two interacting objects are physically in contact with each other.

Tensional force, normal force, force due to air resistance, applied forces and frictional forces are examples of contact forces.

Action-at-a-distance forces (non- contact forces) are forces in which the two interacting objects are not in physical contact which each other, but are able to exert a push or pull despite the physical separation. Gravitational force, electrical force and magnetic force are examples of non- contact forces.

Momentum of a body

It is observed experimentally that the force required to stop a moving object depends on two factors: (i) mass of the body and (ii) its velocity

A body in motion has momentum. *The momentum of a body is defined as the product of its mass and velocity.* If m is the mass of the body and $\vec{v}$, its velocity, the linear momentum of the body is given by $\vec{p} = m\vec{v}$.

Momentum has both magnitude and direction and it is, therefore, a vector quantity. The momentum is measured in terms of $kg\ m\ s^{-1}$ and its dimensional formula is MLT^{-1}.

When a force acts on a body, its velocity changes, consequently, its momentum also changes. The slowly moving bodies have smaller momentum than fast moving bodies of same mass.

If two bodies of unequal masses and velocities have same momentum, then,

$$\vec{p}_1 = \vec{p}_2$$

(i.e) $m_1 \vec{v}_1 = m_2 \vec{v}_2 \Rightarrow \frac{m_1}{m_2} = \frac{\vec{v}_2}{\vec{v}_1}$

Hence for bodies of same momenta, their velocities are inversely proportional to their masses.

2.4.2 Newton's second law of motion

Newton's first law of motion deals with the behaviour of objects on which all existing forces are balanced. Also, it is clear from the first law of motion that a body in motion needs a force to change the direction of motion or the magnitude of velocity or both. This implies that force is such a physical quantity that causes or tends to cause an acceleration.

Newton's second law of motion deals with the behaviour of objects on which all existing forces are not balanced.

According to this law, *the rate of change of momentum of a body is directly proportional to the external force applied on it and the change in momentum takes place in the direction of the force.*

If $\vec{p}$ is the momentum of a body and $\vec{F}$ the external force acting on it, then according to Newton's second law of motion,

$\vec{F} \propto \frac{d\vec{p}}{dt}$ (or) $\vec{F} = k\frac{d\vec{p}}{dt}$ where k is a proportionality constant.

If a body of mass m is moving with a velocity $\vec{v}$ then, its momentum is given by $\vec{p} = m\vec{v}$.

$$\therefore \vec{F} = k\frac{d}{dt}(m\vec{v}) = k\,m\frac{d\vec{v}}{dt}$$

Unit of force is chosen in such a manner that the constant k is equal to unity. (i.e) $k = 1$.

$$\therefore \vec{F} = m\frac{d\vec{v}}{dt} = m\vec{a} \quad \text{where } \vec{a} = \frac{d\vec{v}}{dt} \text{ is the acceleration produced}$$

in the motion of the body.

The force acting on a body is measured by the product of mass of the body and acceleration produced by the force acting on the body. The second law of motion gives us a measure of the force.

The acceleration produced in the body depends upon the inertia of the body (i.e) greater the inertia, lesser the acceleration. *One newton is defined as that force which, when acting on unit mass produces unit acceleration.* Force is a vector quantity. The unit of force is *kg m* s^{-2} or *newton.* Its dimensional formula is MLT^{-2}.

Impulsive force and Impulse of a force

(i) Impulsive Force

An impulsive force is a very great force acting for a very short time on a body, so that the change in the position of the body during the time the force acts on it may be neglected.

(e.g.) The blow of a hammer, the collision of two billiard balls etc.

(ii) Impulse of a force

The impulse J of a constant force F acting for a time t is defined as the product of the force and time.

(i.e) Impulse = Force × time

$J = F \times t$

The impulse of force F acting over a time interval t is defined by the integral,

$$J = \int_0^t F\ dt \qquad ...(1)$$

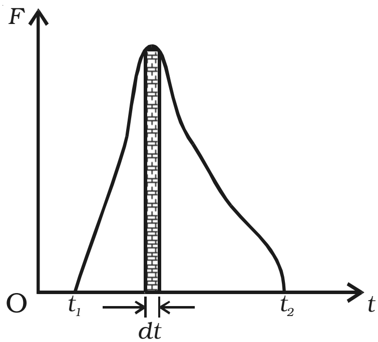


Fig .2.25 Impulse of a force

The impulse of a force, therefore can be visualised as the area under the force versus time graph as shown in Fig. 2.25. When a variable force acting for a short interval of time, then the impulse can be measured as,

$$J = F_{average} \times dt \qquad ...(2)$$

Impulse of a force is a vector quantity and its unit is N s.

Principle of impulse and momentum

By Newton's second law of motion, the force acting on a body = $m\,a$ where m = mass of the body and a = acceleration produced

The impulse of the force = $F \times t = (m\,a)\,t$

If u and v be the initial and final velocities of the body then, $a = \dfrac{(v-u)}{t}$.

Therefore, impulse of the force = $m \times \dfrac{(v-u)}{t} \times t = m(v-u) = mv - mu$

Impulse = final momentum of the body
− initial momentum of the body.

($i.e$) Impulse of the force = Change in momentum

The above equation shows that *the total change in the momentum of a body during a time interval is equal to the impulse of the force acting during the same interval of time. This is called principle of impulse and momentum.*

Examples

(i) A cricket player while catching a ball lowers his hands in the direction of the ball.

If the total change in momentum is brought about in a very short interval of time, the average force is very large according to the equation, $F = \dfrac{mv - mu}{t}$

By increasing the time interval, the average force is decreased. It is for this reason that a cricket player while catching a ball, to increase the time of contact, the player should lower his hand in the direction of the ball , so that he is not hurt.

(ii) A person falling on a cemented floor gets injured more where as a person falling on a sand floor does not get hurt. For the same reason, in wrestling, high jump etc., soft ground is provided.

(iii) The vehicles are fitted with springs and shock absorbers to reduce jerks while moving on uneven or wavy roads.

2.4.3 Newton's third Law of motion

It is a common observation that when we sit on a chair, our body exerts a downward force on the chair and the chair exerts an upward force on our body. There are two forces resulting from this interaction: a force on the chair and a force on our body. These two forces are called action and reaction forces. Newton's third law explains the relation between these action forces. It states that *for every action, there is an equal and opposite reaction.*

(i.e.) whenever one body exerts a certain force on a second body, the second body exerts an equal and opposite force on the first. Newton's third law is sometimes called as the law of action and reaction.

Let there be two bodies 1 and 2 exerting forces on each other. Let the force exerted on the body 1 by the body 2 be $\vec{F}_{12}$ and the force exerted on the body 2 by the body 1 be $\vec{F}_{21}$. Then according to third law, $\vec{F}_{12} = -\vec{F}_{21}$.

One of these forces, say $\vec{F}_{12}$ may be called as the action whereas the other force $\vec{F}_{21}$ may be called as the reaction or vice versa. This implies that we cannot say which is the cause (action) or which is the effect (reaction). It is to be noted that always the action and reaction do not act on the same body; they always act on different bodies. The action and reaction never cancel each other and the forces always exist in pair.

The effect of third law of motion can be observed in many activities in our everyday life. The examples are

(i) When a bullet is fired from a gun with a certain force (action), there is an equal and opposite force exerted on the gun in the backward direction (reaction).

(ii) When a man jumps from a boat to the shore, the boat moves away from him. The force he exerts on the boat (action) is responsible for its motion and his motion to the shore is due to the force of reaction exerted by the boat on him.

(iii) The swimmer pushes the water in the backward direction with a certain force (action) and the water pushes the swimmer in the forward direction with an equal and opposite force (reaction).

(iv) We will not be able to walk if there were no reaction force. In order to walk, we push our foot against the ground. The Earth in turn exerts an equal and opposite force. This force is inclined to the surface of the Earth. The vertical component of this force balances our weight and the horizontal component enables us to walk forward.

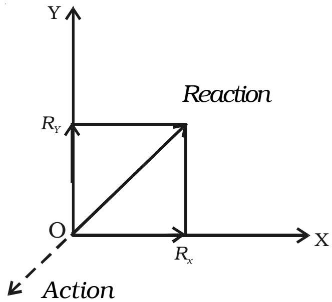


Fig. 2.25a Action and reaction

(v) A bird flies by with the help of its wings. The wings of a bird push air downwards (action). In turn, the air reacts by pushing the bird upwards (reaction).

(vi) When a force exerted directly on the wall by pushing the palm of our hand against it (action), the palm is distorted a little because, the wall exerts an equal force on the hand (reaction).

Law of conservation of momentum

From the principle of impulse and momentum,

impulse of a force, $J = mv - mu$

If $J = 0$ then $mv - mu = 0$ (or) $mv = mu$

(i.e) final momentum = initial momentum

In general, *the total momentum of the system is always a constant (i.e) when the impulse due to external forces is zero, the momentum of the system remains constant. This is known as law of conservation of momentum.*

We can prove this law, in the case of a head on collision between two bodies.

Proof

Consider a body A of mass m_1 moving with a velocity u_1 collides head on with another body B of mass m_2 moving in the same direction as A with velocity u_2 as shown in Fig 2.26.

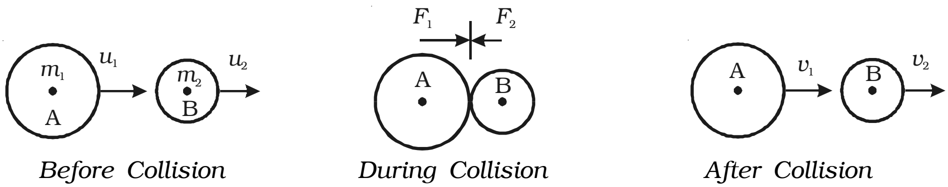


Fig.2.26 Law of conservation of momentum

After collision, let the velocities of the bodies be changed to v_1 and v_2 respectively, and both moves in the same direction. During collision, each body experiences a force.

The force acting on one body is equal in magnitude and opposite in direction to the force acting on the other body. Both forces act for the same interval of time.

Let F_1 be force exerted by A on B (action), F_2 be force exerted by B on A (reaction) and t be the time of contact of the two bodies during collision.

Now, F_1 acting on the body B for a time t, changes its velocity from u_2 to v_2.

$\therefore$ F_1 = mass of the body B × acceleration of the body B

$$= m_2 \times \frac{(v_2 - u_2)}{t} \qquad ...(1)$$

Similarly, F_2 acting on the body A for the same time t changes its velocity from u_1 to v_1

$\therefore$ F_2 = mass of the body A × acceleration of the body A

$$= m_1 \times \frac{(v_1 - u_1)}{t} \qquad ...(2)$$

Then by Newton's third law of motion $F_1 = -F_2$

(i.e) $$m_2 \times \frac{(v_2 - u_2)}{t} = -\, m_1 \times \frac{(v_1 - u_1)}{t}$$

$$m_2\,(v_2 - u_2) = -\, m_1\,(v_1 - u_1)$$

$$m_2\,v_2 - m_2\,u_2 = -\, m_1\,v_1 + m_1\,u_1$$

$$m_1\,u_1 + m_2\,u_2 = m_1\,v_1 + m_2\,v_2 \qquad ...(3)$$

(i.e) total momentum before impact = total momentum after impact.

(i.e) total momentum of the system is a constant.

This proves the law of conservation of linear momentum.

Applications of law of conservation of momentum

The following examples illustrate the law of conservation of momentum.

(i) Recoil of a gun

Consider a gun and bullet of mass m_g and m_b respectively. The gun and the bullet form a single system. Before the gun is fired, both the gun and the bullet are at rest. Therefore the velocities of the gun and bullet are zero. Hence total momentum of the system before firing is $m_g(0) + m_b(0) = 0$

When the gun is fired, the bullet moves forward and the gun recoils backward. Let v_b and v_g are their respective velocities, the total momentum of the bullet – gun system, after firing is $m_b v_b + m_g v_g$

According to the law of conservation of momentum, total momentum before firing is equal to total momentum after firing.

$$\text{(i.e)} \quad 0 = m_b\, v_b + m_g\, v_g \text{ (or) } v_g = -\frac{m_b}{m_g}\, v_b$$

It is clear from this equation, that v_g is directed opposite to v_b. Knowing the values of m_b, m_g and v_b, the recoil velocity of the gun v_g can be calculated.

(ii) Explosion of a bomb

Suppose a bomb is at rest before it explodes. Its momentum is zero. When it explodes, it breaks up into many parts, each part having a particular momentum. A part flying in one direction with a certain momentum, there is another part moving in the opposite direction with the same momentum. If the bomb explodes into two equal parts, they will fly off in exactly opposite directions with the same speed, since each part has the same mass.

Applications of Newton's third law of motion

(i) Apparent loss of weight in a lift

Let us consider a man of mass M standing on a weighing machine placed inside a lift. The actual weight of the man = Mg. This weight (action) is measured by the weighing machine and in turn, the machine offers a reaction R. This reaction offered by the surface of contact on the man is the apparent weight of the man.

Case (i)

When the lift is at rest:

The acceleration of the man = 0

Therefore, net force acting on the man = 0

From Fig. 2.27(i), $R - Mg = 0$ (or) $R = Mg$

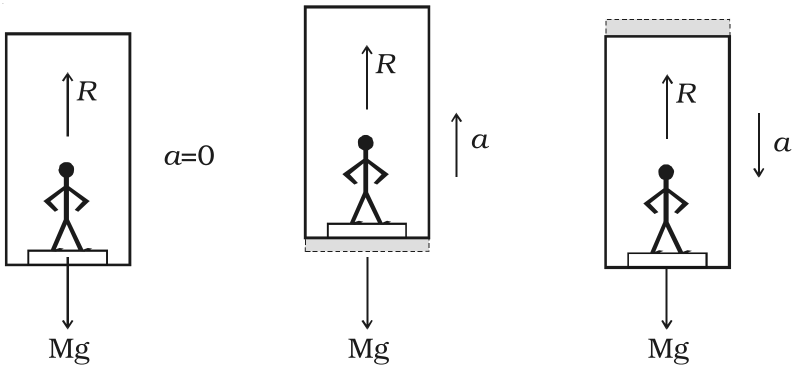


Fig 2.27 Apparent loss of weight in a lift

That is, the apparent weight of the man is equal to the actual weight.

Case (ii)

When the lift is moving uniformly in the upward or downward direction:

For uniform motion, the acceleration of the man is zero. Hence, in this case also the apparent weight of the man is equal to the actual weight.

Case (iii)

When the lift is accelerating upwards:

If a be the upward acceleration of the man in the lift, then the net upward force on the man is $F = Ma$

From Fig 2.27(ii), the net force

$$F = R - Mg = Ma \text{ (or) } R = M(g + a)$$

Therefore, apparent weight of the man is greater than actual weight.

Case (iv)

When the lift is accelerating downwards:

Let a be the downward acceleration of the man in the lift, then the net downward force on the man is $F = Ma$

From Fig. 2.27 (iii), the net force

$$F = Mg - R = Ma \quad \text{(or)} \quad R = M(g - a)$$

Therefore, apparent weight of the man is less than the actual weight.

When the downward acceleration of the man is equal to the acceleration due to the gravity of earth, (i.e) $a = g$

$$\therefore R = M(g - g) = 0$$

Hence, the apparent weight of the man becomes zero. This is known as the weightlessness of the body.

(ii) Working of a rocket and jet plane

The propulsion of a rocket is one of the most interesting examples of Newton's third law of motion and the law of conservation of momentum. The rocket is a system whose mass varies with time. In a rocket, the gases at high temperature and pressure, produced by the combustion of the fuel, are ejected from a nozzle. The reaction of the escaping gases provides the necessary thrust for the launching and flight of the rocket.

From the law of conservation of linear momentum, the momentum of the escaping gases must be equal to the momentum gained by the rocket. Consequently, the rocket is propelled in the forward direction opposite to the direction of the jet of escaping gases. Due to the thrust imparted to the rocket, its velocity and acceleration will keep on increasing. The mass of the rocket and the fuel system keeps on decreasing due to the escaping mass of gases.

2.5 Concurrent forces and Coplanar forces

The basic knowledge of various kinds of forces and motion is highly desirable for engineering and practical applications. The Newton's laws of motion defines and gives the expression for the force. Force is a vector quantity and can be combined according to the rules of vector algebra. A force can be graphically represented by a straight line with an arrow, in which the length of the line is proportional to the magnitude of the force and the arrowhead indicates its direction.

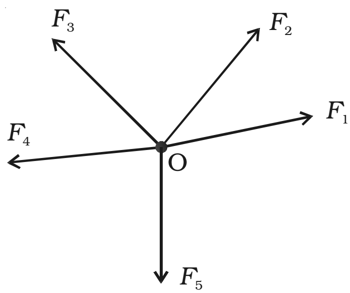


Fig 2.28 Concurrent forces

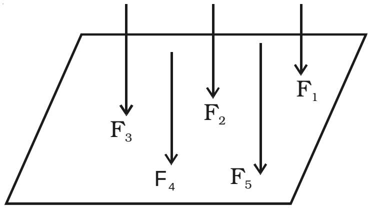


Fig 2.29. Coplanar forces

A force system is said to be concurrent, if the lines of all forces intersect at a common point (Fig 2.28).

A force system is said to be coplanar, if the lines of the action of all forces lie in one plane (Fig 2.29).

2.5.1 Resultant of a system of forces acting on a rigid body

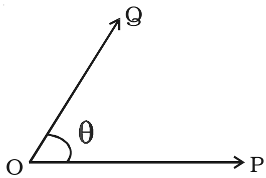
If two or more forces act simultaneously on a rigid body, it is possible to replace the forces by a single force, which will produce the same effect on the rigid body as the effect produced jointly by several forces. This single force is the resultant of the system of forces.

If $\vec{P}$ and $\vec{Q}$ are two forces acting on a body simultaneously in the same direction, their resultant is $\vec{R} = \vec{P} + \vec{Q}$ and it acts in the same direction as that of the forces. If $\vec{P}$ and $\vec{Q}$ act in opposite directions, their resultant $\vec{R}$ is $\vec{R} = \vec{P} \sim \vec{Q}$ and the resultant is in the direction of the greater force.

If the forces $\vec{P}$ and $\vec{Q}$ act in directions which are inclined to each other, their resultant can be found by using parallelogram law of forces and triangle law of forces.

2.5.2 Parallelogram law of forces

If two forces acting at a point are represented in magnitude and direction by the two adjacent sides of a parallelogram, then their resultant is represented in magnitude and direction by the diagonal passing through the point.



Explanation

Consider two forces $\vec{P}$ and $\vec{Q}$ acting at a point *O* inclined at an angle θ as shown in Fig. 2.30.

The forces $\vec{P}$ and $\vec{Q}$ are represented in magnitude and direction by the sides *OA* and *OB* of a parallelogram *OACB* as shown in Fig 2.30.

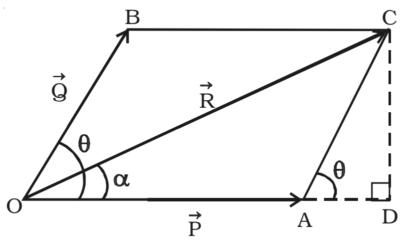


Fig 2.30 Parallelogram law of forces

The resultant $\vec{R}$ of the forces $\vec{P}$ and $\vec{Q}$ is the diagonal *OC* of the parallelogram. The magnitude of the resultant is

$$R = \sqrt{P^2 + Q^2 + 2PQ \cos\theta}$$

The direction of the resultant is $\alpha = \tan^{-1}\left[\frac{Q \sin\theta}{P + Q\cos\theta}\right]$

2.5.3 Triangle law of forces

The resultant of two forces acting at a point can also be found by using triangle law of forces.

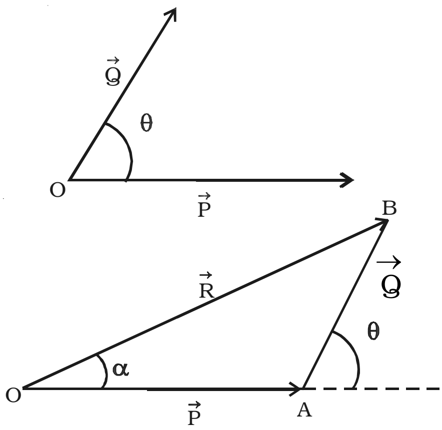


Fig 2.31 Triangle law of forces

If two forces acting at a point are represented in magnitude and direction by the two adjacent sides of a triangle taken in order, then the closing side of the triangle taken in the reversed order represents the resultant of the forces in magnitude and direction.

Forces $\vec{P}$ and $\vec{Q}$ act at an angle θ. In order to find the resultant of $\vec{P}$ and $\vec{Q}$, one can apply the head to tail method, to construct the triangle.

In Fig. 2.31, OA and AB represent $\vec{P}$ and $\vec{Q}$ in magnitude and direction. The closing side OB of the triangle taken in the reversed order represents the resultant $\vec{R}$ of the forces $\vec{P}$ and $\vec{Q}$. The magnitude and the direction of $\vec{R}$ can be found by using sine and cosine laws of triangles.

The triangle law of forces can also be stated as, *if a body is in equilibrium under the action of three forces acting at a point, then the three forces can be completely represented by the three sides of a triangle taken in order.*

If $\vec{P}$, $\vec{Q}$ and $\vec{R}$ are the three forces acting at a point and they are represented by the three sides of a triangle then $\frac{P}{OA} = \frac{Q}{AB} = \frac{R}{OB}$.

2.5.4 Equilibrant

According to Newton's second law of motion, a body moves with a velocity if it is acted upon by a force. When the body is subjected to number of concurrent forces, it moves in a direction of the resultant force. However, if another force, which is equal in magnitude of the resultant but opposite in direction, is applied to a body, the body comes to rest. Hence, *equilibrant of a system of forces is a single force, which acts along with the other forces to keep the body in equilibrium.*

Let us consider the forces F_1. F_2, F_3 and F_4 acting on a body O as shown in Fig. 2.32a. If F is the resultant of all the forces and in order to keep the body at rest, an equal force (known as equilibrant) should act on it in the opposite direction as shown in Fig. 2.32b.

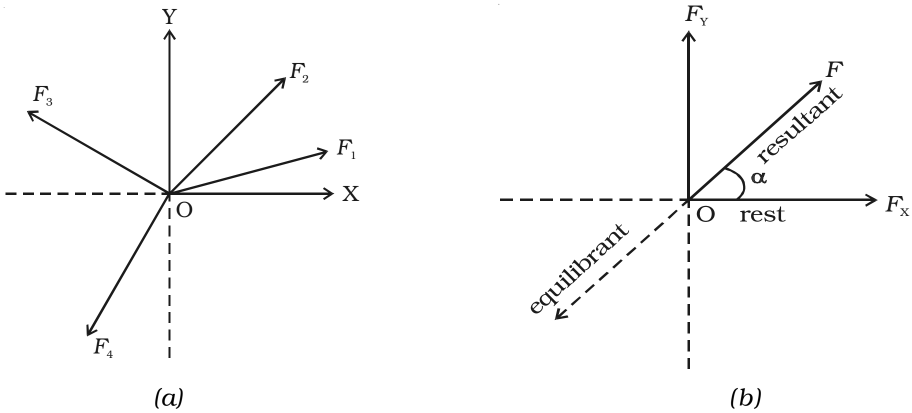


Fig 2.32 Resultant and equilibrant

From Fig. 2.32b, it is found that, resultant = – equilibrant

2.5.5 Resultant of concurrent forces

Consider a body O, which is acted upon by four forces as shown in Fig. 2.33a. Let θ_1, θ_2, θ_3 and θ_4 be the angles made by the forces with respect to X-axis.

Each force acting at O can be replaced by its rectangular components F_{1x} *and* F_{1y}, F_{2x} *and* F_{2y}, .. etc.,

For example, for the force $\vec{F_1}$ making an angle θ_1, its components are, $F_{1x} = F_1 \cos \theta_1$ and $F_{1y} = F_1 \sin \theta_1$

These components of forces produce the same effect on the body as the forces themselves. The algebraic sum of the horizontal components

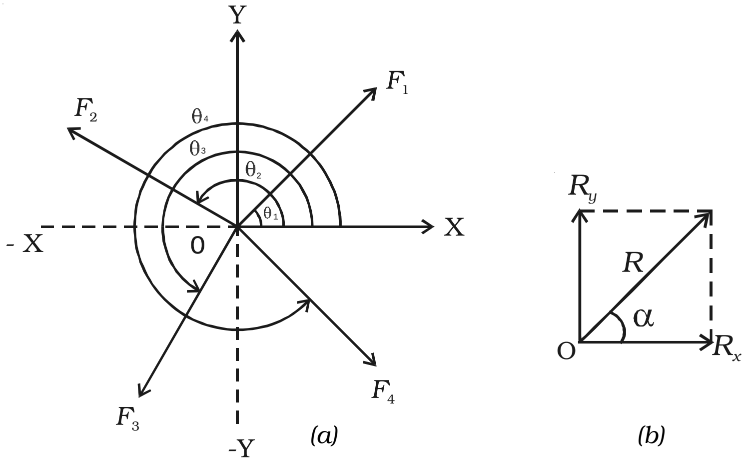


Fig 2.33 Resultant of several concurrent forces

F_{1x}, F_{2x}, F_{3x}, .. gives a single horizontal component R_x

$$\text{(i.e)}\ R_x = F_{1x} + F_{2x} + F_{3x} + F_{4x} = \Sigma F_x$$

Similarly, the algebraic sum of the vertical components F_{1y}, F_{2y}, F_{3y}, .. gives a single vertical component R_y.

$$\text{(i.e)}\ R_y = F_{1y} + F_{2y} + F_{3y} + F_{4y} = \Sigma F_y$$

Now, these two perpendicular components R_x and R_y can be added vectorially to give the resultant $\vec{R}$.

$\therefore$ From Fig. 2.33b, $R^2 = R_x^2 + R_y^2$ (or) $R = \sqrt{R_x^2 + R_y^2}$

and $\tan\alpha = \frac{R_y}{R_x}$ (or) $\alpha = \tan^{-1}\left(\frac{R_y}{R_x}\right)$

2.5.6 Lami's theorem

It gives the conditions of equilibrium for three forces acting at a point. Lami's theorem states that *if three forces acting at a point are in equilibrium, then each of the force is directly proportional to the sine of the angle between the remaining two forces.*

Let us consider three forces $\vec{P}$, $\vec{Q}$ and $\vec{R}$ acting at a point O (Fig 2.34). Under the action of three forces, the point O is at rest, then by Lami's theorem,

$$P \propto \sin\alpha$$

$$Q \propto \sin\beta$$

and $R \propto \sin\gamma$, *then*

$$\frac{P}{\sin\alpha} = \frac{Q}{\sin\beta} = \frac{R}{\sin\gamma} = \text{constant}$$

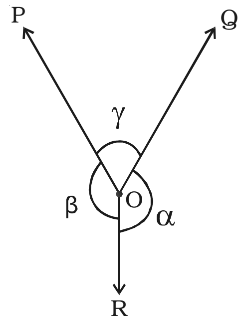


Fig 2.34
Lami's theorem

2.5.7 Experimental verification of triangle law, parallelogram law and Lami's theorem

Two smooth small pulleys are fixed, one each at the top corners of a drawing board kept vertically on a wall as shown in Fig. 2.35. The pulleys should move freely without any friction. A light string is made to pass over both the pulleys. Two slotted weights P and Q (of the order of 50 g) are taken and are tied to the two free ends of the string. Another short string is tied to the centre of the first string at O. A third slotted weight R is attached to the free end of the short string. The weights P, Q and R are adjusted such that the system is at rest.

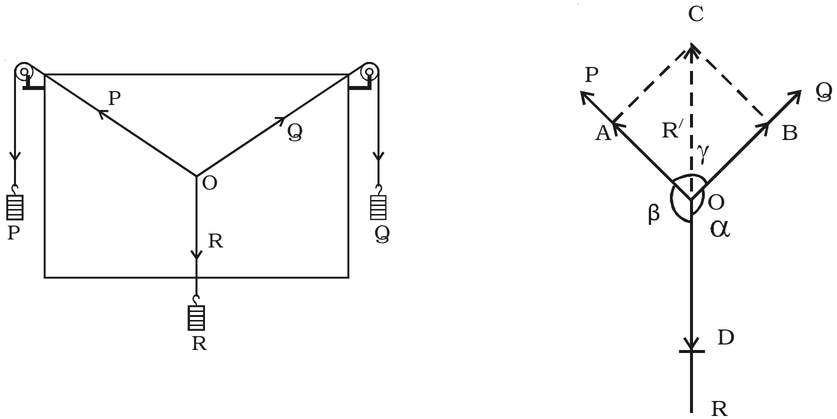


Fig 2.35 Lami's theorem - experimental proof

The point O is in equilibrium under the action of the three forces P, Q and R acting along the strings. Now, a sheet of white paper is held just behind the string without touching them. The common knot O and the directions of OA, OB and OD are marked to represent in magnitude, the three forces P, Q and R on any convenient scale (like 50 g = 1 cm).

The experiment is repeated for different values of P, Q and R and the values are tabulated.

To verify parallelogram law

To determine the resultant of two forces P and Q, a parallelogram OACB is completed, taking OA representing P, OB representing Q and the diagonal OC gives the resultant. The length of the diagonal OC and the angle $\underline{|COD}$ are measured and tabulated (Table 2.2).

OC is the resultant R′ of P and Q. Since O is at rest, this resultant R′ must be equal to the third force R (equilibrant) which acts in the opposite direction. OC = OD. Also, both OC and OD are acting in the opposite direction. $\angle COD$ must be equal to 180°.

If OC = OD and $\angle COD$ = 180°, one can say that parallelogram law of force is verified experimentally.

Table 2.2 Verification of parallelogram law

S.No.	P	Q	R	OA	OB	OD (R)	OC (R′)	$\angle COD$
1.								
2.								
3.								

To verify Triangle Law

According to triangle law of forces, the resultant of P (= OA = BC) and Q (OB) is represented in magnitude and direction by OC which is taken in the reverse direction.

Alternatively, to verify the triangle law of forces, the ratios $\frac{P}{OA}$, $\frac{Q}{OB}$ and $\frac{R'}{OC}$ are calculated and are tabulated (Table 2.3). It will be found out that, all the three ratios are equal, which proves the triangle law of forces experimentally.

Table 2.3 Verification of triangle law

S.No.	P	Q	R′	OA	OB	OC	$\frac{P}{OA}$	$\frac{Q}{OB}$	$\frac{R'}{OC}$
1.									
2.									
3.									

To verify Lami's theorem

To verify Lami's theorem, the angles between the three forces, P, Q and R (i.e) $\angle BOD = \alpha$, $\angle AOD = \beta$ and $\angle AOB = \gamma$ are measured using protractor and tabulated (Table 2.4). The ratios $\frac{P}{\sin\alpha}$, $\frac{Q}{\sin\beta}$ and $\frac{R}{\sin\gamma}$ are calculated and it is found that all the three ratios are equal and this verifies the Lami's theorem.

Table 2.4 Verification of Lami's theorem

S.No.	P	Q	R	α	β	γ	$\frac{P}{\sin\alpha}$	$\frac{Q}{\sin\beta}$	$\frac{R}{\sin\gamma}$
1. 2. 3.									

2.5.8 Conditions of equilibrium of a rigid body acted upon by a system of concurrent forces in plane

(i) If an object is in equilibrium under the action of three forces, the resultant of two forces must be equal and opposite to the third force. Thus, the line of action of the third force must pass through the point of intersection of the lines of action of the other two forces. In other words, the system of three coplanar forces in equilibrium, must obey parallelogram law, triangle law of forces and Lami's theorem. This condition ensures the absence of translational motion in the system.

(ii) The algebraic sum of the moments about any point must be equal to zero. $\Sigma M = 0$ (i.e) the sum of clockwise moments about any point must be equal to the sum of anticlockwise moments about the same point. This condition ensures, the absence of rotational motion.

2.6 Uniform circular motion

The revolution of the Earth around the Sun, rotating fly wheel, electrons revolving around the nucleus, spinning top, the motion of a fan blade, revolution of the moon around the Earth etc. are some examples of circular motion. In all the above cases, the bodies or particles travel in a circular path. So, it is necessary to understand the motion of such bodies.

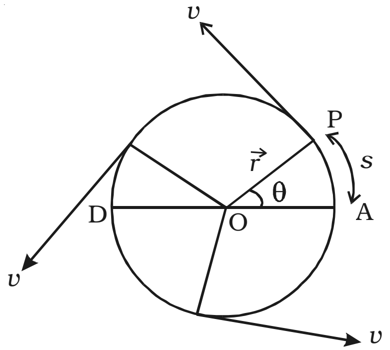


Fig. 2.36 Uniform circular motion

When a particle moves on a circular path with a constant speed, then its motion is known as uniform circular motion in a plane. The magnitude of velocity in circular motion remains constant but the direction changes continuously.

Let us consider a particle of mass *m* moving with a velocity *v* along the circle of radius *r* with centre O as shown in Fig 2.36. P is the position of the particle at a given instant of time such that the radial line OP makes an angle θ with the reference line DA. The magnitude of the velocity remains constant, but its direction changes continuously. The linear velocity always acts tangentially to the position of the particle (i.e) in each position, the linear velocity $\vec{v}$ is perpendicular to the radius vector $\vec{r}$.

2.6.1 Angular displacement

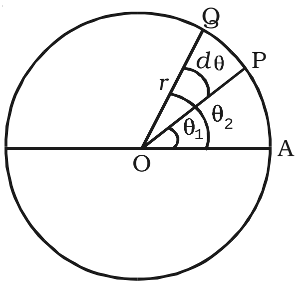


Fig. 2.37 Angular displacement

Let us consider a particle of mass *m* moving along the circular path of radius *r* as shown in Fig. 2.37. Let the initial position of the particle be A. P and Q are the positions of the particle at any instants of time *t* and *t* + *dt* respectively. Suppose the particle traverses a distance *ds* along the circular path in time interval *dt*. During this interval, it moves through an angle $d\theta = \theta_2 - \theta_1$. *The angle swept by the radius vector at a given time is called the angular displacement of the particle.*

If *r* be the radius of the circle, then the angular displacement is given by $d\theta = \frac{ds}{r}$. The angular displacement is measured in terms of radian.

2.6.2 Angular velocity

The rate of change of angular displacement is called the angular velocity of the particle.

Let $d\theta$ be the angular displacement made by the particle in time dt , then the angular velocity of the particle is $\omega = \frac{d\theta}{dt}$. Its unit is rad s^{-1} and dimensional formula is T^{-1}.

For one complete revolution, the angle swept by the radius vector is 360° or 2π radians. If T is the time taken for one complete revolution, known as period, then the angular velocity of the particle is $\omega = \frac{\theta}{t} = \frac{2\pi}{T}$.

If the particle makes n revolutions per second, then $\omega = 2\pi\left(\frac{1}{T}\right) = 2\pi n$ where $n = \frac{1}{T}$ is the frequency of revolution.

2.6.3 Relation between linear velocity and angular velocity

Let us consider a body P moving along the circumference of a circle of radius r with linear velocity v and angular velocity ω as shown in Fig. 2.38. Let it move from P to Q in time dt and $d\theta$ be the angle swept by the radius vector.

Let PQ = ds, be the arc length covered by the particle moving along the circle, then the angular displacement $d\theta$ is expressed as $d\theta = \frac{ds}{r}$. But $ds = v\,dt$

$$\therefore\ d\theta = \frac{v\,dt}{r} \quad \text{(or)} \quad \frac{d\theta}{dt} = \frac{v}{r}$$

(i.e) Angular velocity $\omega = \frac{v}{r}$ or $v = \omega r$

In vector notation, $\vec{v} = \vec{\omega} \times \vec{r}$

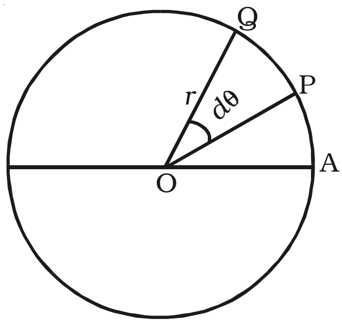


Fig 2.38 Relation between linear velocity and angular velocity

Thus, for a given angular velocity ω, the linear velocity v of the particle is directly proportional to the distance of the particle from the centre of the circular path (i.e) *for a body in a uniform circular motion, the angular velocity is the same for all points in the body but linear velocity is different for different points of the body.*

2.6.4 Angular acceleration

If the angular velocity of the body performing rotatory motion is non-uniform, then the body is said to possess angular acceleration.

The rate of change of angular velocity is called angular acceleration.

If the angular velocity of a body moving in a circular path changes from ω_1 to ω_2 in time t then its angular acceleration is

$$\alpha = \frac{d\omega}{dt} = \frac{d}{dt}\left(\frac{d\theta}{dt}\right) = \frac{d^2\theta}{dt^2} = \frac{\omega_2 - \omega_1}{t}.$$

The angular acceleration is measured in terms of rad s^{-2} and its dimensional formula is T^{-2}.

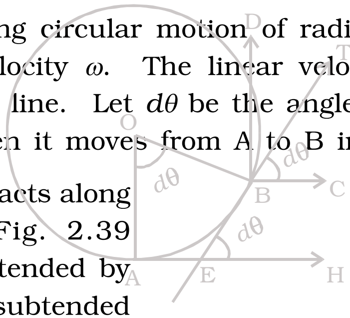
2.6.5 *Relation between linear acceleration and angular acceleration*

If dv is the small change in linear velocity in a time interval dt then linear acceleration is $a = \frac{dv}{dt} = \frac{d}{dt}(r\omega) = r\frac{d\omega}{dt} = r\alpha$.

2.6.6 *Centripetal acceleration*

The speed of a particle performing uniform circular motion remains constant throughout the motion but its velocity changes continuously due to the change in direction (i.e) *the particle executing uniform circular motion is said to possess an acceleration.*

Consider a particle executing circular motion of radius r with linear velocity v and angular velocity ω. The linear velocity of the particle acts along the tangential line. Let $d\theta$ be the angle described by the particle at the centre when it moves from A to B in time dt.



At A and B, linear velocity v acts along AH and BT respectively. In Fig. 2.39 $\angle AOB = d\theta = \angle HET$ ($\because$ angle subtended by the two radii of a circle = angle subtended by the two tangents).

The velocity v at B of the particle makes an angle $d\theta$ with the line BC and hence it is resolved horizontally as $v \cos d\theta$ along BC and vertically as $v \sin d\theta$ along BD.

Fig 2.39 Centripetal acceleration

$\therefore$ The change in velocity along the horizontal direction = $v \cos d\theta - v$

If $d\theta$ is very small, $\cos d\theta = 1$

$\therefore$ Change in velocity along the horizontal direction = $v - v = 0$

(i.e) there is no change in velocity in the horizontal direction.

The change in velocity in the vertical direction (i.e along AO) is

$dv = v \sin d\theta - 0 = v \sin d\theta$

If $d\theta$ is very small, $\sin d\theta = d\theta$

$\therefore$ The change in velocity in the vertical direction (i.e) along radius of the circle

$$dv = v.d\theta \quad ...(1)$$

But, acceleration $a = \frac{dv}{dt} = \frac{v\, d\theta}{dt} = v\omega$ ...(2)

where $\omega = \frac{d\theta}{dt}$ is the angular velocity of the particle.

We know that $v = r\,\omega$ ...(3)

From equations (2) and (3),

$$a = r\omega\ \omega = r\omega^2 = \frac{v^2}{r} \quad ...(4)$$

Hence, *the acceleration of the particle producing uniform circular motion is equal to* $\frac{v^2}{r}$ *and is along AO (i.e) directed towards the centre of the circle. This acceleration is directed towards the centre of the circle along the radius and perpendicular to the velocity of the particle. This acceleration is known as centripetal or radial or normal acceleration.*

2.6.7 Centripetal force

According to Newton's first law of motion, a body possesses the property called directional inertia (i.e) the inability of the body to change its direction. This means that without the application of an external force, the direction of motion can not be changed. Thus when a body is moving along a circular path, some force must be acting upon it, which continuously changes the body from its straight-line path (Fig 2.40). It makes clear that the applied force should have no component in the direction of the motion of the body or the force must act at every

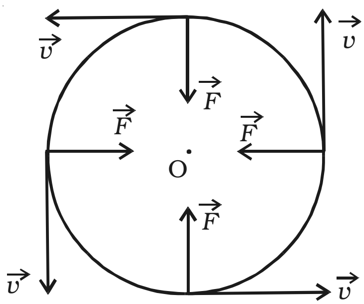


Fig 2.40 Centripetal force

point perpendicular to the direction of motion of the body. This force, therefore, must act along the radius and should be directed towards the centre.

Hence *for circular motion, a constant force should act on the body, along the radius towards the centre and perpendicular to the velocity of the body. This force is known as centripetal force.*

If m is the mass of the body, then the magnitude of the centripetal force is given by

$$F = \text{mass} \times \text{centripetal acceleration}$$

$$= m\left(\frac{v^2}{r}\right) = \frac{mv^2}{r} = m\,(r\omega^2)$$

Examples

Any force like gravitational force, frictional force, electric force, magnetic force etc. may act as a centripetal force. Some of the examples of centripetal force are :

(i) In the case of a stone tied to the end of a string whirled in a circular path, the centripetal force is provided by the tension in the string.

(ii) When a car takes a turn on the road, the frictional force between the tyres and the road provides the centripetal force.

(iii) In the case of planets revolving round the Sun or the moon revolving round the earth, the centripetal force is provided by the gravitational force of attraction between them

(iv) For an electron revolving round the nucleus in a circular path, the electrostatic force of attraction between the electron and the nucleus provides the necessary centripetal force.

2.6.8 Centrifugal reaction

According to Newton's third law of motion, for every action there is an equal and opposite reaction. *The equal and opposite reaction to the centripetal force is called centrifugal reaction, because it tends to take the body away from the centre.* In fact, the centrifugal reaction is a pseudo or apparent force, acts or assumed to act because of the acceleration of the rotating body.

In the case of a stone tied to the end of the string is whirled in a circular path, not only the stone is acted upon by a force (centripetal force) along the string towards the centre, but the stone also exerts an equal and opposite force on the hand (centrifugal force) away from the centre, along the string. On releasing the string, the tension disappears and the stone flies off tangentially to the circular path along a straight line as enuciated by Newton's first law of motion.

When a car is turning round a corner, the person sitting inside the car experiences an outward force. It is because of the fact that no centripetal force is supplied by the person. Therefore, to avoid the outward force, the person should exert an inward force.

2.6.9 Applications of centripetal forces

(i) Motion in a vertical circle

Let us consider a body of mass m tied to one end of the string which is fixed at O and it is moving in a vertical circle of radius r about the point O as shown in Fig. 2.41. The motion is circular but is not uniform, since the body speeds up while coming down and slows down while going up.

Suppose the body is at P at any instant of time t, the tension T in the string always acts towards O.

The weight mg of the body at P is resolved along the string as $mg \cos\theta$ which acts outwards and $mg \sin\theta$, perpendicular to the string.

When the body is at P, the following forces acts on it along the string.

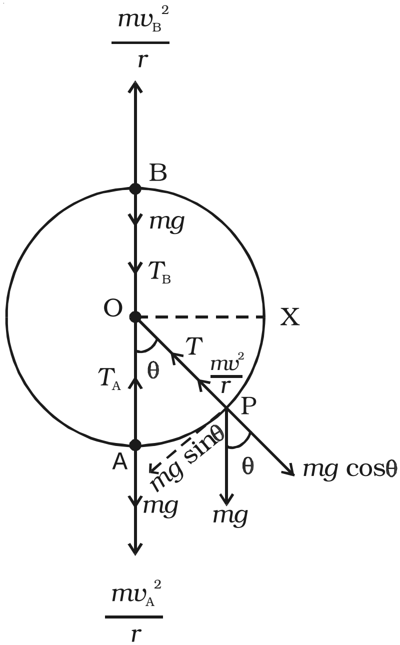


Fig. 2.41 Motion of a body in a vertical circle

(i) $mg \cos\theta$ acts along OP (outwards)

(ii) tension T acts along PO (inwards)

Net force on the body at P acting along $PO = T - mg \cos\theta$

This must provide the necessary centripetal force $\frac{mv^2}{r}$.

Therefore, $T - mg \cos\theta = \frac{mv^2}{r}$

$$T = mg \cos\theta + \frac{mv^2}{r} \qquad \ldots(1)$$

At the lowest point A of the path, $\theta = 0^{o}$, $cos\ 0^{o} = 1$ then

from equation (1), $T_A = mg + \frac{mv_A^2}{r}$ ...(2)

At the highest point of the path, i.e. at B, $\theta = 180^{o}$. Hence $cos\ 180^{o} = -1$

$\therefore$ from equation (1), $T_B = -mg + \frac{mv_B^2}{r} = \frac{mv_B^2}{r} - mg$

$$T_B = m\left(\frac{v_B^2}{r} - g\right) \quad ...(3)$$

If $T_B > 0$, then the string remains taut while if $T_B < 0$, the string slackens and it becomes impossible to complete the motion in a vertical circle.

If the velocity v_B is decreased, the tension T_B in the string also decreases, and becomes zero at a certain minimum value of the speed called *critical velocity*. Let v_C be the minimum value of the velocity, then at $v_B = v_C$, $T_B = 0$. Therefore from equation (3),

$$\frac{mv_C^2}{r} - mg = 0 \quad \text{(or)} \quad v_C^2 = rg$$

(i.e) $v_C = \sqrt{rg}$ ...(4)

If the velocity of the body at the highest point B is below this critical velocity, the string becomes slack and the body falls downwards instead of moving along the circular path. In order to ensure that the velocity v_B at the top is not lesser than the critical velocity $\sqrt{rg}$, the minimum velocity v_A at the lowest point should be in such a way that v_B should be $\sqrt{rg}$. (i.e) the motion in a vertical circle is possible only if $v_B \geq \sqrt{rg}$.

The velocity v_A of the body at the bottom point A can be obtained by using law of conservation of energy. When the stone rises from A to B, i.e through a height $2r$, its potential energy increases by an amount equal to the decrease in kinetic energy. Thus,

(Potential energy at A + Kinetic energy at A) =
(Potential energy at B + Kinetic energy at B)

(i.e.) $0 + \frac{1}{2} m v_A^2 = mg\ (2r) + \frac{1}{2} m v_B^2$

Dividing by $\frac{m}{2}$, $v_A^2 = v_B^2 + 4gr$ ...(5)

But from equation (4), $v_B^2 = gr \qquad (\because v_B = v_C)$

$\therefore$ Equation (5) becomes, $v_A^2 = gr + 4gr$ (or) $v_A = \sqrt{5gr}$ ...(6)

Substituting v_A from equation (6) in (2),

$$T_A = mg + \frac{m(5gr)}{r} = mg + 5mg = 6\ mg \qquad ...(7)$$

While rotating in a vertical circle, the stone must have a velocity greater than $\sqrt{5gr}$ or tension greater than 6mg at the lowest point, so that its velocity at the top is greater than $\sqrt{gr}$ or tension ≥ 0.

An aeroplane while looping a vertical circle must have a velocity greater than $\sqrt{5gr}$ at the lowest point, so that its velocity at the top is greater than $\sqrt{gr}$. In that case, pilot sitting in the aeroplane will not fall.

(ii) Motion on a level circular road

When a vehicle goes round a level curved path, it should be acted upon by a centripetal force. While negotiating the curved path, the wheels of the car have a tendency to leave the curved path and regain the straight-line path. Frictional force between the tyres and the road opposes this tendency of the wheels. This frictional force, therefore, acts towards the centre of the circular path and provides the necessary centripetal force.

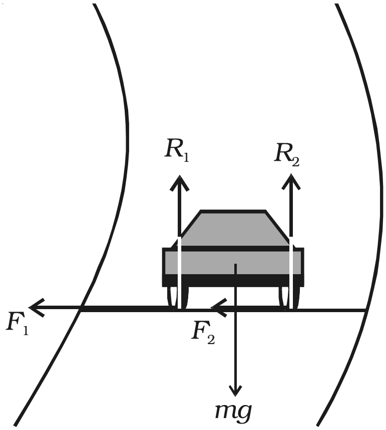


Fig. 2.42 Vehicle on a level circular road

In Fig. 2.42, weight of the vehicle mg acts vertically downwards. R_1, R_2 are the forces of normal reaction of the road on the wheels. As the road is level (horizontal), R_1, R_2 act vertically upwards. Obviously,

$$R_1 + R_2 = mg \qquad ...(1)$$

Let μ * be the coefficient of friction between the tyres and the

***Friction** : Whenever a body slides over another body, a force comes into play between the two surfaces in contact and this force is known as frictional force. The frictional force always acts in the opposite direction to that of the motion of the body. The frictional force depends on the normal reaction. (Normal reaction is a perpendicular reactional force that acts on the body at the point of contact due to its own weight) (i.e) Frictional force α normal reaction F α R (or) F = μR where μ is a proportionality constant and is known as the coefficient of friction. The coefficient of friction depends on the nature of the surface.*

road, F_1 and F_2 be the forces of friction between the tyres and the road, directed towards the centre of the curved path.

$$\therefore\ F_1 = \mu R_1 \text{ and } F_2 = \mu R_2 \qquad ...(2)$$

If v is velocity of the vehicle while negotiating the curve, the centripetal force required $= \frac{mv^2}{r}$.

As this force is provided only by the force of friction.

$$\therefore \quad \frac{mv^2}{r} \le (F_1 + F_2)$$

$$\le (\mu R_1 + \mu R_2)$$

$$\le \mu (R_1 + R_2)$$

$$\therefore \quad \frac{mv^2}{r} \le \mu mg \quad (\because R_1 + R_2 = mg)$$

$$v^2 \le \mu rg$$

$$v \le \sqrt{\mu rg}$$

Hence the maximum velocity with which a car can go round a level curve without skidding is $v = \sqrt{\mu rg}$. The value of v depends on radius r of the curve and coefficient of friction μ between the tyres and the road.

(iii) Banking of curved roads and tracks

When a car goes round a level curve, the force of friction between the tyres and the road provides the necessary centripetal force. If the frictional force, which acts as centripetal force and keeps the body moving along the circular road is not enough to provide the necessary centripetal force, the car will skid. In order to avoid skidding, while going round a curved path the outer edge of the road is raised above the level of the inner edge. This is known as banking of curved roads or tracks.

Bending of a cyclist round a curve

A cyclist has to bend slightly towards the centre of the circular track in order to take a safe turn without slipping.

Fig. 2.43 shows a cyclist taking a turn towards his right on a circular path of radius r. Let m be the mass of the cyclist along with the bicycle and v, the velocity. When the cyclist negotiates the curve, he bends inwards from the vertical, by an angle θ. Let R be the reaction

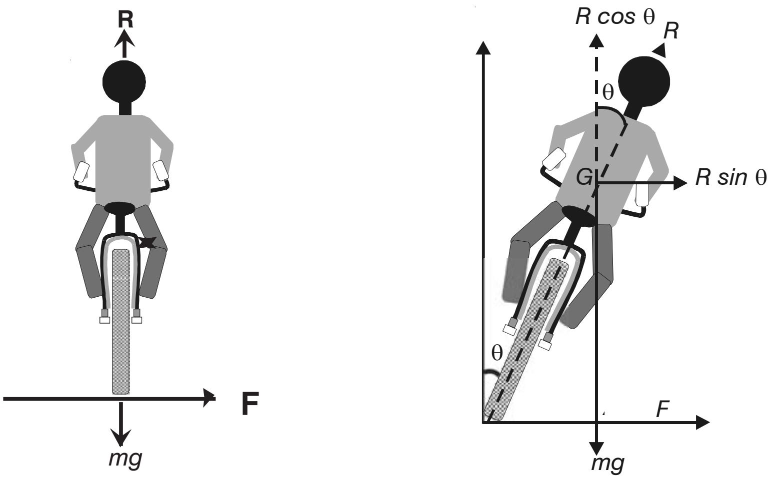


Fig 2.43 Bending of a cyclist in a curved road

of the ground on the cyclist. The reaction R may be resolved into two components: (i) the component $R \sin \theta$, acting towards the centre of the curve providing necessary centripetal force for circular motion and (ii) the component $R \cos \theta$, balancing the weight of the cyclist along with the bicycle.

(i.e) $$R \sin \theta = \frac{mv^2}{r} \quad \ldots(1)$$

and $$R \cos \theta = mg \quad \ldots(2)$$

Dividing equation (1) by (2), $$\frac{R \sin \theta}{R \cos \theta} = \frac{\frac{mv^2}{r}}{mg}$$

$$\tan \theta = \frac{v^2}{rg} \quad \ldots(3)$$

Thus for less bending of cyclist (i.e for θ to be small), the velocity v should be smaller and radius r should be larger.

For a banked road (Fig. 2.44), let h be the elevation of the outer edge of the road above the inner edge and l be the width of the road then,

$$\sin \theta = \frac{h}{l} \quad \ldots(4)$$

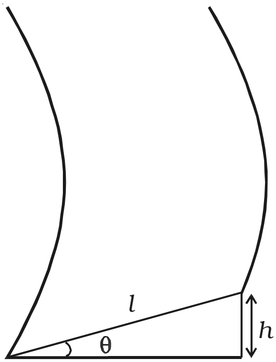


Fig 2.44 Banked road

For small values of θ, *sin* θ = *tan* θ

Therefore from equations (3) and (4)

$$\tan\theta = \frac{h}{l} = \frac{v^2}{rg} \quad ...(5)$$

Obviously, a road or track can be banked correctly only for a particular speed of the vehicle. Therefore, the driver must drive with a particular speed at the circular turn. If the speed is higher than the desired value, the vehicle tends to slip outward at the turn but then the frictional force acts inwards and provides the additional centripetal force. Similarly, if the speed of the vehicle is lower than the desired speed it tends to slip inward at the turn but now the frictional force acts outwards and reduces the centripetal force.

Condition for skidding

When the centripetal force is greater than the frictional force, skidding occurs. If μ is the coefficient of friction between the road and tyre, then the limiting friction (frictional force) is $f = \mu R$ where normal reaction $R = mg$

$$\therefore f = \mu\ (mg)$$

Thus for skidding,

Centripetal force > Frictional force

$$\frac{mv^2}{r} > \mu\ (mg)$$

$$\frac{v^2}{rg} > \mu$$

But $$\frac{v^2}{rg} = \tan\theta$$

$$\therefore \tan\theta > \mu$$

(i.e) when the tangent of the angle of banking is greater than the coefficient of friction, skidding occurs.

2.7 Work

The terms work and energy are quite familiar to us and we use them in various contexts. In everyday life, the term work is used to refer to any form of activity that requires the exertion of mental or muscular efforts. In physics, *work is said to be done by a force or* *against the direction of the force, when the point of application of the force moves towards or against the direction of the force.* If no displacement takes place, no work is said to be done. Therefore for work to be done, two essential conditions should be satisfied:

(i) *a force must be exerted*

(ii) *the force must cause a motion or displacement*

If a particle is subjected to a force F and if the particle is displaced by an infinitesimal displacement ds, the work done dw by the force is $dw = \vec{F} \cdot \vec{ds}$.

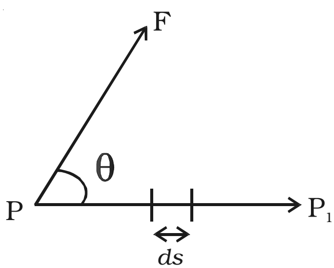


Fig. 2.45 Work done by a force

The magnitude of the above dot product is $F \cos\theta \, ds$.

(i.e) $dw = F\,ds \cos\theta = (F \cos\theta)\, ds$ where θ = angle between $\vec{F}$ *and* $\vec{ds}$. *(Fig. 2.45)*

Thus, *the work done by a force during an infinitesimal displacement is equal to the product of the displacement ds and the component of the force F cos θ in the direction of the displacement.*

Work is a scalar quantity and has magnitude but no direction.

The work done by a force when the body is displaced from position P to P_1 can be obtained by integrating the above equation,

$$W = \int dw = \int (F \cos\theta)\, ds$$

Work done by a constant force

When the force F acting on a body has a constant magnitude and acts at a constant angle θ from the straight line path of the particle as shown as Fig. 2.46, then,

$$W = F \cos\theta \int_{s_1}^{s_2} ds = F \cos\theta (s_2 - s_1)$$

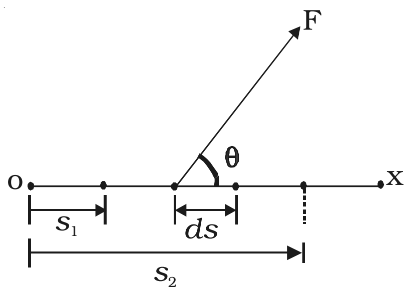


Fig. 2.46 Work done by a constant force

The graphical representation of work done by a constant force is shown in Fig 2.47.

$$W = F \cos\theta\, (s_2 - s_1) = \text{area ABCD}$$

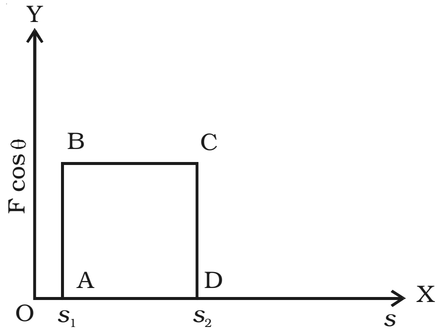


Fig.2.47 Graphical representation of work done by a constant force

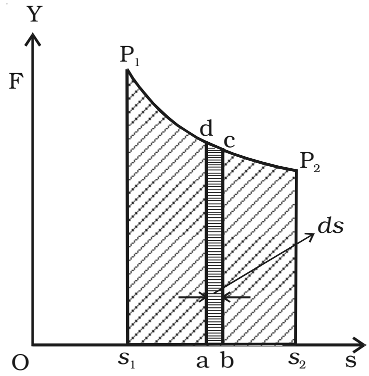


Fig 2.48 Work done by a variable force

Work done by a variable force

If the body is subjected to a varying force F and displaced along X axis as shown in Fig 2.48, work done

$dw = F\cos\theta . ds$ = area of the small element abcd.

$\therefore$ The total work done when the body moves from s_1 to s_2 is

$\Sigma\, dw = W$ = area under the curve P_1P_2 = area $S_1\, P_1\, P_2\, S_2$

The unit of work is joule. *One joule is defined as the work done by a force of one newton when its point of application moves by one metre along the line of action of the force.*

Special cases

(i) When $\theta = 0$, the force F is in the same direction as the displacement s.

$$\therefore \text{Work done, } W = F\, s \cos 0 = F\, s$$

(ii) When $\theta = 90°$, the force under consideration is normal to the direction of motion.

$$\therefore \text{Work done, } W = F\, s \cos 90° = 0$$

For example, if a body moves along a frictionless horizontal surface, its weight and the reaction of the surface, both normal to the surface, do no work. Similarly, when a stone tied to a string is whirled around in a circle with uniform speed, the centripetal force continuously changes the direction of motion. Since this force is always normal to the direction of motion of the object, it does no work.

(iii) When $\theta = 180°$, the force F is in the opposite direction to the displacement.

$\therefore$ Work done (W) = $F\ s \cos 180° = -F\ s$

(eg.) The frictional force that slows the sliding of an object over a surface does a negative work.

A *positive work* can be defined as the *work done by a force* and a *negative work* as the *work done against a force.*

2.8 Energy

Energy can be defined as the capacity to do work. Energy can manifest itself in many forms like mechanical energy, thermal energy, electric energy, chemical energy, light energy, nuclear energy, etc.

The energy possessed by a body due to its position or due to its motion is called mechanical energy.

The mechanical energy of a body consists of potential energy and kinetic energy.

2.8.1 Potential energy

The potential energy of a body is the energy stored in the body by virtue of its position or the state of strain. Hence water stored in a reservoir, a wound spring, compressed air, stretched rubber chord, etc, possess potential energy.

Potential energy is given by the amount of work done by the force acting on the body, when the body moves from its given position to some other position.

Expression for the potential energy

Let us consider a body of mass m, which is at rest at a height h above the ground as shown in Fig 2.49. The work done in raising the body from the ground to the height h is stored in the body as its potential energy and when the body falls to the ground, the same amount of work can be got back from it. Now, in order to lift the body vertically up, a force mg equal to the weight of the body should be applied.

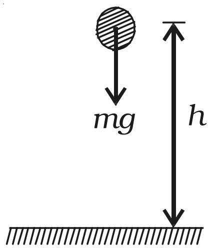


Fig. 2.49
Potential energy

When the body is taken vertically up through a height h, then work done, W = Force × displacement

$$\therefore\ W = mg \times h$$

This work done is stored as potential energy in the body

$$\therefore\ E_P = mgh$$

2.8.2 Kinetic energy

The kinetic energy of a body is the energy possessed by the body by virtue of its motion. It is measured by the amount of work that the body can perform against the impressed forces before it comes to rest. A falling body, a bullet fired from a rifle, a swinging pendulum, etc. possess kinetic energy.

A body is capable of doing work if it moves, but in the process of doing work its velocity gradually decreases. The amount of work that can be done depends both on the magnitude of the velocity and the mass of the body. A heavy bullet will penetrate a wooden plank deeper than a light bullet of equal size moving with equal velocity.

Expression for Kinetic energy

Let us consider a body of mass m moving with a velocity v in a straightline as shown in Fig. 2.50. Suppose that it is acted upon by a constant force F resisting its motion, which produces retardation a (decrease in acceleration is known as retardation). Then

$$F = mass \times retardation = -\,ma \qquad ...(1)$$

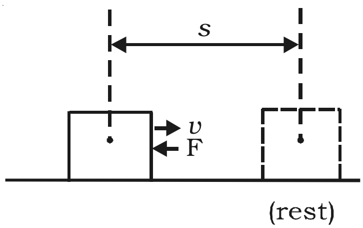


Fig. 2.50 Kinetic energy

Let dx be the displacement of the body before it comes to rest.

But the retardation is

$$a = \frac{dv}{dt} = \frac{dv}{dx} \times \frac{dx}{dt} = \frac{dv}{dx} \times v \qquad ...(2)$$

where $\frac{dx}{dt} = v$ is the velocity of the body

Substituting equation (2) in (1), $F = -\,mv\,\frac{dv}{dx}$ ...(3)

Hence the work done in bringing the body to rest is given by,

$$W = \int F.dx = -\int_v^0 mv.\frac{dv}{dx}.dx = -m\int_v^0 v\,dv \qquad ...(4)$$

$$W = -m\left[\frac{v^2}{2}\right]_v^0 = \frac{1}{2}\,mv^2$$

This work done is equal to kinetic energy of the body.

$\therefore$ Kinetic energy $E_k = \frac{1}{2} mv^2$

2.8.3 Principle of work and energy (work – energy theorem)

Statement

The work done by a force acting on the body during its displacement is equal to the change in the kinetic energy of the body during that displacement.

Proof

Let us consider a body of mass m acted upon by a force F and moving with a velocity v along a path as shown in Fig. 2.51. At any instant, let P be the position of the body from the origin O. Let θ be the angle made by the direction of the force with the tangential line drawn at P.

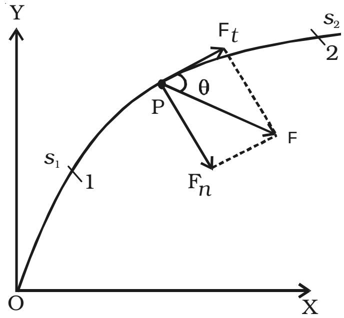


Fig. 2.51
Work–energy theorem

The force F can be resolved into two rectangular components :

(i) $F_t = F \cos\theta$, tangentially and

(ii) $F_n = F \sin\theta$, normally at P.

But $F_t = ma_t$ ...(1)

where a_t is the acceleration of the body in the tangential direction

$$\therefore \quad F \cos\theta = ma_t \qquad \text{...(2)}$$

$$\text{But } a_t = \frac{dv}{dt} \qquad \text{...(3)}$$

$\therefore$ substituting equation (3) in (2),

$$F \cos\theta = m\frac{dv}{dt} = m\frac{dv}{ds}\cdot\frac{ds}{dt} \qquad \text{...(4)}$$

$$F \cos\theta \; ds = mv \; dv \qquad \text{...(5)}$$

where ds is the small displacement.

Let v_1 and v_2 be the velocities of the body at the positions 1 and 2 and the corresponding distances be s_1 and s_2.

Integrating the equation (5),

$$\int_{s_1}^{s_2} (F \cos\theta)\, ds = \int_{v_1}^{v_2} mv \; dv \qquad \text{...(6)}$$

But $\int_{s_1}^{s_2} (F \cos \theta)\, ds = W_{1\to2}$ ...(7)

where $W_{1\to2}$ is the work done by the force

From equation (6) and (7),

$$W_{1\to2} = \int_{v_1}^{v_2} mv\, dv$$

$$= m \left[\frac{v^2}{2}\right]_{v_1}^{v_2} = \frac{mv_2^2}{2} - \frac{mv_1^2}{2} \quad ...(8)$$

Therefore work done

= final kinetic energy – initial kinetic energy

= change in kinetic energy

This is known as Work–energy theorem.

2.8.4 Conservative forces and non-conservative forces

Conservative forces

If the work done by a force in moving a body between two positions is independent of the path followed by the body, then such a force is called as a conservative force.

Examples : force due to gravity, spring force and elastic force.

The work done by the conservative forces depends only upon the initial and final position of the body.

(*i.e.*) $\oint \vec{F} \cdot \vec{dr} = 0$

The work done by a conservative force around a closed path is zero.

Non conservative forces

Non-conservative force is the force, which can perform some resultant work along an arbitrary closed path of its point of application.

The work done by the non-conservative force depends upon the path of the displacement of the body (i.e.) $\oint \vec{F} \cdot \vec{dr} \neq 0$

(e.g) Frictional force, viscous force, etc.

2.8.5 *Law of conservation of energy*

The law states that, if a body or system of bodies is in motion under a conservative system of forces, the sum of its kinetic energy and potential energy is constant.

Explanation

From the principle of work and energy,

Work done = change in the kinetic energy

(i.e) $W_{1\to2} = E_{k2} - E_{k1}$ ...(1)

If a body moves under the action of a conservative force, work done is stored as potential energy.

$W_{1\to2} = -(E_{P2} - E_{P1})$ ...(2)

Work done is equal to negative change of potential energy. Combining the equation (1) and (2),

$Ek_2 - Ek_1 = -(E_{P2} - E_{P1})$ (or) $E_{P1} + E_{k1} = E_{P2} + E_{k2}$ ...(3)

which means that *the sum of the potential energy and kinetic energy of a system of particles remains constant during the motion under the action of the conservative forces.*

2.8.6 Power

It is defined as the rate at which work is done.

$$power = \frac{work\ done}{time}$$

Its unit is watt and dimensional formula is ML^2T^{-3}.

Power is said to be one watt, when one joule of work is said to be done in one second.

If dw is the work done during an interval of time dt then,

$$\text{power} = \frac{dw}{dt} \quad ...(1)$$

But $dw = (F \cos \theta)\, ds$ ...(2)

where θ is the angle between the direction of the force and displacement. $F \cos \theta$ is component of the force in the direction of the small displacement ds.

Substituting equation (2) in (1) power = $\frac{(F \cos \theta)\, ds}{dt}$

$$= (F \cos \theta) \frac{ds}{dt} = (F \cos \theta)\, v \qquad \left(\because \frac{ds}{dt} = v\right)$$

$$\therefore \text{power} = (F \cos \theta)\, v$$

If F and v are in the same direction, then

power = $F\, v \cos 0 = F\, v$ = Force × velocity

It is also represented by the dot product of F and v.

(i.e) $P = \vec{F} \cdot \vec{v}$

2.9 Collisions

A collision between two particles is said to occur if they physically strike against each other or if the path of the motion of one is influenced by the other. In physics, the term collision does not necessarily mean that a particle actually strikes. In fact, two particles may not even touch each other and yet they are said to collide if one particle influences the motion of the other.

When two bodies collide, each body exerts a force on the other. The two forces are exerted simultaneously for an equal but short interval of time. According to Newton's third law of motion, each body exerts an equal and opposite force on the other at each instant of collision. During a collision, the two fundamental conservation laws namely, the law of conservation of momentum and that of energy are obeyed and these laws can be used to determine the velocities of the bodies after collision.

Collisions are divided into two types : (i) elastic collision and (ii) inelastic collision

2.9.1 Elastic collision

If the kinetic energy of the system is conserved during a collision, it is called an elastic collision. (i.e) The total kinetic energy before collision and after collision remains unchanged. The collision between subatomic particles is generally elastic. The collision between two steel or glass balls is nearly elastic. In elastic collision, the linear momentum and kinetic energy of the system are conserved.

Elastic collision in one dimension

If the two bodies after collision move in a straight line, the collision is said to be of one dimension.

Consider two bodies A and B of masses m_1 and m_2 moving along the same straight line in the same direction with velocities u_1 and u_2 respectively as shown in Fig. 2.54. Let us assume that u_1 is greater than u_2. The bodies A and B suffer a head on collision when they strike and continue to move along the same straight line with velocities v_1 and v_2 respectively.

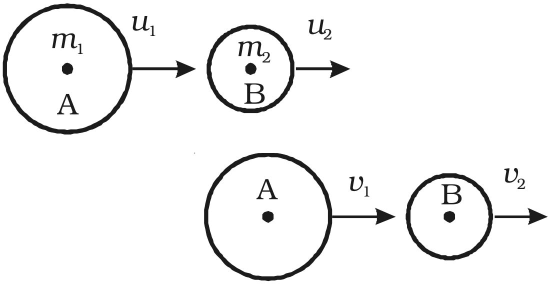


Fig 2.54 Elastic collision in one dimension

From the law of conservation of linear momentum,

Total momentum before collision = Total momentum after collision

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2 \quad ...(1)$$

Since the kinetic energy of the bodies is also conserved during the collision

Total kinetic energy before collision = Total kinetic energy after collision

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 \quad ...(2)$$

$$m_1u_1^2 - m_1v_1^2 = m_2v_2^2 - m_2u_2^2 \quad ...(3)$$

From equation (1) $m_1(u_1 - v_1) = m_2(v_2 - u_2)$ ...(4)

Dividing equation (3) by (4),

$$\frac{u_1^2 - v_1^2}{u_1 - v_1} = \frac{v_2^2 - u_2^2}{v_2 - u_2} \quad (or) \quad u_1 + v_1 = u_2 + v_2$$

$$(u_1 - u_2) = (v_2 - v_1) \quad ...(5)$$

Equation (5) shows that in an elastic one-dimensional collision, the relative velocity with which the two bodies approach each other before collision is equal to the relative velocity with which they recede from each other after collision.

From equation (5), $v_2 = u_1 - u_2 + v_1$ ...(6)

Substituting v_2 in equation (4),

$$m_1 (u_1 - v_1) = m_2 (v_1 - u_2 + u_1 - u_2)$$

$$m_1u_1 - m_1v_1 = m_2u_1 - 2m_2u_2 + m_2v_1$$

$$(m_1 + m_2)v_1 = m_1u_1 - m_2u_1 + 2m_2u_2$$

$$(m_1 + m_2)v_1 = u_1 (m_1 - m_2) + 2m_2u_2$$

$$v_1 = u_1 \left[\frac{m_1 - m_2}{m_1 + m_2}\right] + \frac{2m_2u_2}{(m_1 + m_2)} \quad ...(7)$$

Similarly, $$v_2 = \frac{2m_1u_1}{(m_1 + m_2)} + \frac{u_2(m_2 - m_1)}{(m_1 + m_2)} \quad ...(8)$$

Special cases

Case (i) : If the masses of colliding bodies are equal, i.e. $m_1 = m_2$

$v_1 = u_2$ and $v_2 = u_1$ ...(9)

After head on elastic collision, the velocities of the colliding bodies are mutually interchanged.

Case (ii) : If the particle *B* is initially at rest, (i.e) $u_2 = 0$ then

$$v_1 = \frac{(m_A - m_B)}{(m_A + m_B)} u_A \quad ...(10)$$

and $$v_2 = \frac{2m_A}{(m_A + m_B)} u_1 \quad ...(11)$$

2.9.2 Inelastic collision

During a collision between two bodies if there is a loss of kinetic energy, then the collision is said to be an inelastic collision. Since there is always some loss of kinetic energy in any collision, collisions are generally inelastic. In inelastic collision, the linear momentum is conserved but the energy is not conserved. If two bodies stick together, after colliding, the collision is perfectly inelastic but it is a special case of inelastic collision called plastic collision. (eg) a bullet striking a block of wood and being embedded in it. The loss of kinetic energy usually results in the form of heat or sound energy.

Let us consider a simple situation in which the inelastic head on collision between two bodies of masses m_A and m_B takes place. Let the colliding bodies be initially move with velocities u_1 and u_2. After collision both bodies stick together and moves with common velocity v.

Total momentum of the system before collision = $m_A u_1 + m_B u_2$

Total momentum of the system after collision =
mass of the composite body × common velocity = $(m_A + m_B)\, v$

By law of conservation of momentum

$$m_A u_1 + m_B u_2 = (m_A + m_B)\, v \quad \text{(or)} \quad v = \frac{m_A u_A + m_B u_B}{m_A + m_B}$$

Thus, knowing the masses of the two bodies and their velocities before collision, the common velocity of the system after collision can be calculated.

If the second particle is initially at rest i.e. $u_2 = 0$ then

$$v = \frac{m_A u_A}{(m_A + m_B)}$$

kinetic energy of the system before collision

$$E_{K1} = \frac{1}{2} m_A u_A^2 \qquad [\because\ u_2 = 0]$$

and kinetic energy of the system after collision

$$E_{K2} = \frac{1}{2}(m_A + m_B) v^2$$

Hence,

$$\frac{E_{K2}}{E_{K1}} = \frac{\textit{kinetic energy after collision}}{\textit{kinetic energy before collision}}$$

$$= \frac{(m_A + m_B) v^2}{m_A u_A^2}$$

Substituting the value of v in the above equation,

$$\frac{E_{K2}}{E_{K1}} = \frac{m_A}{m_A + m_B} \quad \text{(or)} \quad \frac{E_{K2}}{E_{K1}} < 1$$

It is clear from the above equation that in a perfectly inelastic collision, the kinetic energy after impact is less than the kinetic energy before impact. The loss in kinetic energy may appear as heat energy.

Solved Problems

2.1 The driver of a car travelling at 72 kmph observes the light 300 m ahead of him turning red. The traffic light is timed to remain red for 20 s before it turns green. If the motorist wishes to passes the light without stopping to wait for it to turn green, determine (i) the required uniform acceleration of the car (ii) the speed with which the motorist crosses the traffic light.

Data : $u = 72\ kmph = 72 \times \frac{5}{18}\ m\,s^{-1} = 20\ m\,s^{-1}$; $S = 300\ m$;

$t = 20\ s$; $a = ?$; $v = ?$

Solution : i) $s = ut + \frac{1}{2}at^2$

$$300 = (20 \times 20) + \frac{1}{2}a(20)^2$$

$$a = -0.5\ m\,s^{-2}$$

ii) $v = u + at = 20 - 0.5 \times 20 = 10\ m\,s^{-1}$

2.2 A stone is dropped from the top of the tower 50 m high. At the same time another stone is thrown up from the foot of the tower with a velocity of 25 m s^{-1} . At what distance from the top and after how much time the stones cross each other?

Data: *Height of the tower = 50 m* $u_1 = 0$; $u_2 = 25\ m\,s^{-1}$

Let s_1 and s_2 be the distances travelled by the two stones at the time of crossing (t). Therefore $s_1 + s_2 = 50m$

$s_1 = ?$; $t = ?$

Solution : *For I stone :* $s_1 = \frac{1}{2}g\,t^2$

For II stone : $s_2 = u_2 t - \frac{1}{2}g\,t^2$

$$s_2 = 25\,t - \frac{1}{2}g\,t^2$$

Therefore, $s_1 + s_2 = 50 = \frac{1}{2}gt^2 + 25\,t - \frac{1}{2}gt^2$

$t = 2$ *seconds*

$$s_1 = \frac{1}{2}gt^2 = \frac{1}{2}(9.8)(2)^2 = 19.6\ m$$

2.3 A boy throws a ball so that it may just clear a wall 3.6m high. The boy is at a distance of 4.8 m from the wall. The ball was found to hit the ground at a distance of 3.6m on the other side of the wall. Find the least velocity with which the ball can be thrown.

Data : *Range of the ball* $= 4.8 + 3.6 = 8.4m$

Height of the wall $= 3.6m$

$u = ?\ ;\ \theta = ?$

Solution : *The top of the wall AC must lie on the path of the projectile.*

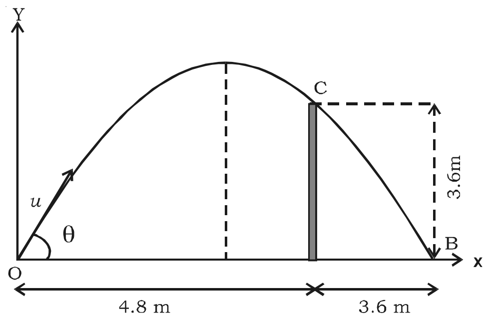
The equation of the projectile is $y = x\tan\theta - \dfrac{gx^2}{2u^2\cos^2\theta}$ ...(1)

The point C (x = 4.8m, y = 3.6m) lies on the trajectory.

Substituting the known values in (1),

$$3.6 = 4.8\tan\theta - \frac{g \times (4.8)^2}{2u^2\cos^2\theta} \qquad ...(2)$$

The range of the projectile is $R = \dfrac{u^2 \sin 2\theta}{g} = 8.4$ ...(3)



From (3), $\dfrac{u^2}{g} = \dfrac{8.4}{\sin 2\theta}$ ...(4)

Substituting (4) in (2),

$$3.6 = (4.8)\tan\theta - \frac{(4.8)^2}{2\cos^2\theta} \times \frac{\sin 2\theta}{(8.4)}$$

$$3.6 = (4.8)\tan\theta - \frac{(4.8)^2}{2\cos^2\theta} \times \frac{2\sin\theta\cos\theta}{(8.4)}$$

$$3.6 = (4.8)\tan\theta - (2.7429)\tan\theta$$

$$3.6 = (2.0571)\tan\theta$$

$$\tan\theta = \frac{3.6}{2.0571} = 1.7500$$

$$\theta = \tan^{-1}[1.75] = 60^\circ 15'$$

Substituting the value of θ *in (4),*

$$u^2 = \frac{8.4 \times g}{\sin 2\theta} = \frac{8.4 \times 9.8}{\sin 2(60^\circ 15')} = 95.5399$$

$$u = 9.7745 \text{ m s}^{-1}$$

2.4 Prove that for a given velocity of projection, the horizontal range is same for two angles of projection α and (90° – α).

The horizontal range is given by, $R = \dfrac{u^2 \sin 2\theta}{g}$ ...(1)

When θ = α,

$$R_1 = \frac{u^2 \sin 2\alpha}{g} \quad ...(2)$$

When θ = (90° − α),

$$R_2 = \frac{u^2 \sin 2(90^0 - \alpha)}{g} = \frac{u^2[2\sin(90^0-\alpha)\cos(90^0-\alpha)]}{g} \quad ...(3)$$

But $\sin(90^0 - \alpha) = \cos\alpha$; $\cos(90^0 - \alpha) = \sin\alpha$

$$R_2 = \frac{u^2(2\sin\alpha\cos\alpha)}{g} = \frac{u^2 \sin 2\alpha}{g} \quad ...(4)$$

From (2) and (4), it is seen that at both angles α *and (90 − α), the horizontal range remains the same.*

2.5 The pilot of an aeroplane flying horizontally at a height of 2000 m with a constant speed of 540 kmph wishes to hit a target on the ground. At what distance from the target should release the bomb to hit the target?

Data : *Initial velocity of the bomb in the horizontal is the same as that of the air plane.*

Initial velocity of the bomb in the horizontal

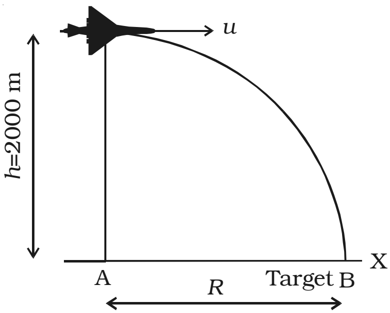
direction = *540 kmph* = $540 \times \frac{5}{18}$ $m\,s^{-1}$ = $150\ m\,s^{-1}$

Initial velocity in the vertical direction (u) = 0 ; vertical distance (s) = 2000 m ; time of flight t = ?

Solution : *From equation of motion,*

$$s = ut + \frac{1}{2}at^2$$

Substituting the known values,



$$2000 = 0 \times t + \frac{1}{2} \times 9.8 \times t^2$$

$$2000 = 4.9t^2 \qquad \text{(or)}$$

$$t = \sqrt{\frac{2000}{4.9}} = 20.20\ s$$

∴ *horizontal range = horizontal velocity × time of flight*

$$= 150 \times 20.20 = 3030\ m$$

2.6 Two equal forces are acting at a point with an angle of 60° between them. If the resultant force is equal to $20\sqrt{3}$ N, find the magnitude of each force.

Data : *Angle between the forces,* $\theta = 60°$; *Resultant* $R = 20\sqrt{3}\ N$

$P = Q = P\ (say) = \ ?$

Solution : $R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta}$

$$= \sqrt{P^2 + P^2 + 2P.P\cos 60°}$$

$$= \sqrt{2P^2 + 2P^2 \cdot \frac{1}{2}} \qquad = P\sqrt{3}$$

$$20\sqrt{3} = P\sqrt{3}$$

$$P = 20\ N$$

2.7 If two forces $F_1 = 20\ kN$ and $F_2 = 15\ kN$ act on a particle as shown in figure, find their resultant by triangle law.

Data : $F_1 = 20$ kN; $F_2 = 15$ k N; R=?

Solution : *Using law of cosines,*

$$R^2 = P^2 + Q^2 - 2PQ\cos(180 - \theta)$$

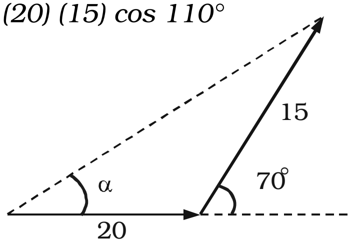
$$R^2 = 20^2 + 15^2 - 2(20)(15)\cos 110^\circ$$

$$\therefore R = 28.813\ kN.$$

Using law of sines,

$$\frac{R}{\sin 110} = \frac{15}{\sin \alpha}$$

$$\therefore \quad \alpha = 29.3^\circ$$



2.8 Two forces act at a point in directions inclined to each other at 120°. If the bigger force is 5 kg *wt* and their resultant is at right angles to the smaller force, find the resultant and the smaller force.

Data : *Bigger force = 5 kg wt*

Angle made by the resultant with the smaller force = 90°

Resultant = ? *Smaller force = ?*

Solution : *Let the forces P and Q are acting along OA and OD where ∠AOD =120°*

Complete the parallelogram OACD and join OC. OC therefore which represents the resultant which is perpendicular to OA.

In ΔOAC

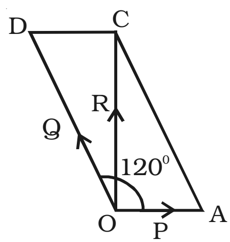
$$\angle OCA = \angle COD = 30^\circ$$

$$\angle AOC = 90^\circ$$

Therefore $\angle OAC = 60^\circ$

(i.e) $$\frac{P}{\sin 30} = \frac{Q}{\sin 90} = \frac{R}{\sin 60}$$

Since Q = 5 kg. wt.



$$P = \frac{5\sin 30}{\sin 90} = 2.5\ kg\ wt$$

$$R = \frac{5\sin 60^\circ}{\sin 90^\circ} = \frac{5\sqrt{3}}{2}\ kg\ wt$$

2.9 Determine analytically the magnitude and direction of the resultant of the following four forces acting at a point.

(i) 10 kN pull N 30° E; (ii) 20 kN push S 45° W;

(iii) 5 kN push N 60° W; (iv) 15 kN push S 60° E.

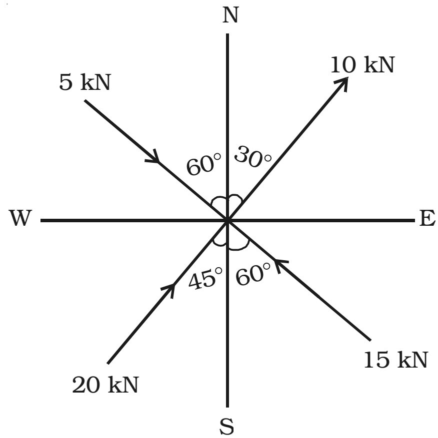
Data : $F_1 = 10\ kN$;

$F_2 = 20\ kN$;

$F_3 = 5\ kN$;

$F_4 = 15\ kN$;

$R = ?$; $\alpha = ?$



Solution : *The various forces acting at a point are shown in figure.*

Resolving the forces horizontally, we get

$\Sigma F_x = 10 \sin 30° + 5 \sin 60° + 20 \sin 45° - 15 \sin 60°$

$= 10.48\ kN$

Similarly, resolving forces vertically, we get

$\Sigma F_y = 10 \cos 30° - 5 \cos 60° + 20 \cos 45° + 15 \cos 60°$

$= 27.8\ kN$

Resultant $R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2}$

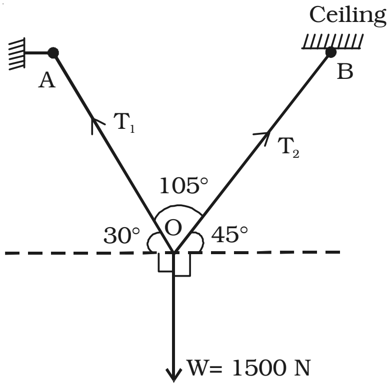
$= \sqrt{(10.48)^2 + (27.8)^2}$

$= 29.7\ kN$

$$\tan \alpha = \frac{\Sigma F_y}{\Sigma F_x} = \frac{27.8}{10.48} = 2.65$$

$\alpha = 69.34^o$

2.10 A machine weighing 1500 N is supported by two chains attached to some point on the machine. One of these ropes goes to a nail in the wall and is inclined at 30° to the horizontal and other goes to the hook in ceiling and is inclined at 45° to the horizontal. Find the tensions in the two chains.



Data : *W = 1500 N, Tensions in the strings = ?*

Solution : *The machine is in equilibrium under the following forces :*

(i) W (weight of the machine) acting vertically down ;

(ii) Tension T_1 in the chain OA;

(iii) Tension T_2 in the chain OB.

Now applying Lami's theorem at O, we get

$$\frac{T_1}{\sin\ (90^o + 45^o)} = \frac{T_2}{\sin\ (90^o + 30^o)} = \frac{T_3}{\sin\ 105^o}$$

$$\frac{T_1}{\sin\ 135^o} = \frac{T_2}{\sin\ 120^o} = \frac{1500}{\sin\ 105^o}$$

$$T_1 = \frac{1500 \times \sin\ 135^o}{\sin\ 105^o} = 1098.96\ N$$

$$T_2 = \frac{1500 \times \sin\ 120^o}{\sin\ 105^o} = 1346.11\ N$$

2.11 The radius of curvature of a railway line at a place when a train is moving with a speed of 72 kmph is 1500 m. If the distance between the rails is 1.54 m, find the elevation of the outer rail above the inner rail so that there is no side pressure on the rails.

Data : *r = 1500 m ; v = 72 kmph= 20 m s^{-1} ; l = 1.54 m ; h = ?*

Solution : $\tan\theta = \dfrac{h}{l} = \dfrac{v^2}{rg}$

Therefore $h = \dfrac{lv^2}{rg} = \dfrac{1.54 \times (20)^2}{1500 \times 9.8} = 0.0419$ m

2.12 A truck of weight 2 tonnes is slipped from a train travelling at 9 kmph and comes to rest in 2 minutes. Find the retarding force on the truck.

Data : $m = 2 \text{ tonne} = 2 \times 1000 \text{ kg} = 2000 \text{ kg}$

$$v_1 = 9 \text{ kmph} = 9 \times \frac{5}{18} = \frac{5}{2} \text{ m s}^{-1}; \quad v_2 = 0$$

Solution : *Let R newton be the retarding force.*

By the momentum - impulse theorem ,

$(mv_1 - mv_2) = Rt$ (or) $m v_1 - Rt = mv_2$

$2000 \times \frac{5}{2} - R \times 120 = 2000 \times 0$ (or) $5000 - 120 R = 0$

$R = 41.67 \text{ N}$

2.13 A body of mass 2 kg initially at rest is moved by a horizontal force of 0.5N on a smooth frictionless table. Obtain the work done by the force in 8 s and show that this is equal to change in kinetic energy of the body.

Data : $M = 2 \text{ kg}$; $F = 0.5 \text{ N}$; $t = 8 \text{ s}$; $W = ?$

Solution : $\therefore$ *Acceleration produced (a)* $= \frac{F}{m} = \frac{0.5}{2} = 0.25 \text{ m s}^{-2}$

The velocity of the body after 8s $= a \times t = 0.25 \times 8 = 2 \text{ m s}^{-1}$

The distance covered by the body in 8 s $= S = ut + \frac{1}{2} at^2$

$$S = (0 \times 8) + \frac{1}{2}(0.25)(8)^2 = 8 \text{ m}$$

$\therefore$ *Work done by the force in 8 s =*

$$\text{Force} \times \text{distance} = 0.5 \times 8 = 4 \text{ J}$$

Initial kinetic energy $= \frac{1}{2} m (0)^2 = 0$

Final kinetic energy $= \frac{1}{2} mv^2 = \frac{1}{2} \times 2 \times (2)^2 = 4 \text{ J}$

$\therefore$ *Change in kinetic energy = Final K.E. – Initial K.E* $= 4 - 0 = 4 \text{ J}$

The work done is equal to the change in kinetic energy of the body.

2.14 A body is thrown vertically up from the ground with a velocity of 39.2 m s^{-1}. At what height will its kinetic energy be reduced to one – fourth of its original kinetic energy.

Data : *v = 39.2 m s^{-1} ; h = ?*

Solution : *When the body is thrown up, its velocity decreases and hence potential energy increases.*

Let h be the height at which the potential energy is reduced to one – fourth of its initial value.

(i.e) loss in kinetic energy = gain in potential energy

$$\frac{3}{4} \times \frac{1}{2} m v^2 = mg h$$

$$\frac{3}{4} \times \frac{1}{2} (39.2)^2 = 9.8 \times h$$

$$h = 58.8 \text{ m}$$

2.15 A 10 g bullet is fired from a rifle horizontally into a 5 kg block of wood suspended by a string and the bullet gets embedded in the block. The impact causes the block to swing to a height of 5 cm above its initial level. Calculate the initial velocity of the bullet.

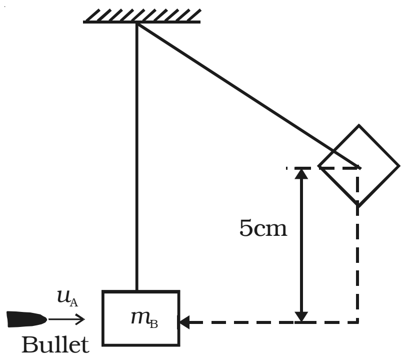
Data : *Mass of the bullet = m_A = 10 g = 0.01 kg*

Mass of the wooden block = m_B = 5 kg

Initial velocity of the bullet before impact = u_A = ?

Initial velocity of the block before impact = u_B = 0

Final velocity of the bullet and block = v



Solution : *By law of conservation of linear momentum,*

$$m_A u_A + m_B u_B = (m_A + m_B)\, v$$

$$(0.01)u_A + (5 \times 0) = (0.01 + 5)\, v$$

$$\text{(or)} \quad v = \left(\frac{0.01}{5.01}\right) u_A = \frac{u_A}{501} \qquad \text{...(1)}$$

Applying the law of conservation of mechanical energy,

KE of the combined mass = PE at the highest point

$$\text{(or)} \quad \frac{1}{2}(m_A + m_B)\, v^2 = (m_A + m_B)\, gh \qquad \text{...(2)}$$

From equation (1) and (2),

$$\frac{u_A^2}{(501)^2} = 2gh \ \text{(or)}\ u_A = \sqrt{2.46 \times 10^5} = 496.0\ m\ s^{-1}$$

3. Dynamics of Rotational Motion

3.1 Centre of mass

Every body is a collection of large number of tiny particles. In translatory motion of a body, every particle experiences equal displacement with time; therefore the motion of the whole body may be represented by a particle. But when the body rotates or vibrates during translatory motion, then its motion can be represented by a point on the body that moves in the same way as that of a single particle subjected to the same external forces would move. *A point in the system at which whole mass of the body is supposed to be concentrated is called centre of mass of the body.* Therefore, if a system contains two or more particles, its translatory motion can be described by the motion of the centre of mass of the system.

3.1.1 Centre of mass of a two-particle system

Let us consider a system consisting of two particles of masses m_1 and m_2. P_1 and P_2 are their positions at time t and r_1 and r_2 are the corresponding distances from the origin O as shown in Fig. 3.1. Then the velocity and acceleration of the particles are,

$$v_1 = \frac{dr_1}{dt} \quad ...(1)$$

$$a_1 = \frac{dv_1}{dt} \quad ...(2)$$

$$v_2 = \frac{dr_2}{dt} \quad ...(3)$$

$$a_2 = \frac{dv_2}{dt} \quad ...(4)$$

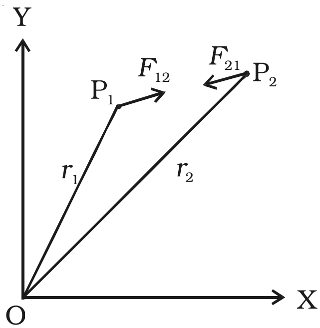


Fig 3.1 – Centre of mass

The particle at P_1 experiences two forces :

(i) a force F_{12} due to the particle at P_2 and

(ii) force F_{1e} , the external force due to some particles external to the system.

If F_1 is the resultant of these two forces,

$$F_1 = F_{12} + F_{1e} \qquad ...(5)$$

Similarly, the net force F_2 acting on the particle P_2 is,

$$F_2 = F_{21} + F_{2e} \qquad ...(6)$$

where F_{21} is the force exerted by the particle at P_1 on P_2

By using Newton's second law of motion,

$$F_1 = m_1 a_1 \qquad ...(7)$$

and $F_2 = m_2 a_2$ ...(8)

Adding equations (7) and (8), $m_1 a_1 + m_2 a_2 = F_1 + F_2$

Substituting F_1 and F_2 from (5) and (6)

$$m_1 a_1 + m_2 a_2 = F_{12} + F_{1e} + F_{21} + F_{2e}$$

By Newton's third law, the internal force F_{12} exerted by particle at P_2 on the particle at P_1 is equal and opposite to F_{21}, the force exerted by particle at P_1 on P_2.

(i.e) $F_{12} = - F_{21}$ ...(9)

$$\therefore F = F_{1e} + F_{2e} \qquad ...(10)$$

$$[\because m_1 a_1 + m_2 a_2 = F]$$

where F is the net external force acting on the system.

The total mass of the system is given by,

$$M = m_1 + m_2 \qquad ...(11)$$

Let the net external force F acting on the system produces an acceleration a_{CM} called the acceleration of the centre of mass of the system

By Newton's second law, for the system of two particles,

$$F = M\, a_{CM} \qquad ...(12)$$

From (10) and (12), $M\, a_{CM} = m_1 a_1 + m_2 a_2$ ...(13)

Let R_{CM} be the position vector of the centre of mass.

$$\therefore a_{CM} = \frac{d^2(R_{CM})}{dt^2} \qquad ...(14)$$

From (13) and (14),

$$\frac{d^2 R_{CM}}{dt^2} = \left(\frac{1}{M}\right)\left(m_1 \frac{d^2 r_1}{dt^2} + m_2 \frac{d^2 r_2}{dt^2}\right)$$

$$\frac{d^2R_{CM}}{dt^2} = \frac{1}{M}\left(\frac{d^2}{dt^2}(m_1r_1 + m_2r_2)\right)$$

$$\therefore R_{CM} = \frac{1}{M}\left(m_1r_1 + m_2r_2\right)$$

$$R_{CM} = \frac{m_1r_1 + m_2r_2}{m_1 + m_2} \quad ...(15)$$

This equation gives the position of the centre of mass of a system comprising two particles of masses m_1 and m_2

If the masses are equal ($m_1 = m_2$), then the position vector of the centre of mass is,

$$R_{CM} = \frac{r_1 + r_2}{2} \quad ...(16)$$

which means that the centre of mass lies exactly in the middle of the line joining the two masses.

3.1.2 Centre of mass of a body consisting of n particles

For a system consisting of n particles with masses m_1, m_2, m_3 ... m_n with position vectors r_1, r_2, r_3...r_n, the total mass of the system is,

$$M = m_1 + m_2 + m_3 + ... + m_n$$

The position vector R_{CM} of the centre of mass with respect to origin O is given by

$$R_{CM} = \frac{m_1r_1 + m_2r_2 ..... + m_nr_n}{m_1 + m_2 ..... + m_n} = \frac{\sum_{i=1}^{n} m_ir_i}{\sum_{i=1}^{n} m_i} = \frac{\sum_{i=1}^{n} m_ir_i}{M}$$

The x coordinate and y coordinate of the centre of mass of the system are

$$x = \frac{m_1x_1 + m_2x_2 + ...m_nx_n}{m_1 + m_2 + ...m_n} \quad \text{and} \quad y = \frac{m_1y_1 + m_2y_2 + ...m_ny_n}{m_1 + m_2 + ...m_n}$$

Example for motion of centre of mass

Let us consider the motion of the centre of mass of the Earth and moon system (Fig 3.2). The moon moves round the Earth in a circular orbit and the Earth moves round the Sun in an elliptical orbit. It is more correct to say that the Earth and the moon both move in circular orbits about their common centre of mass in an elliptical orbit round the Sun.

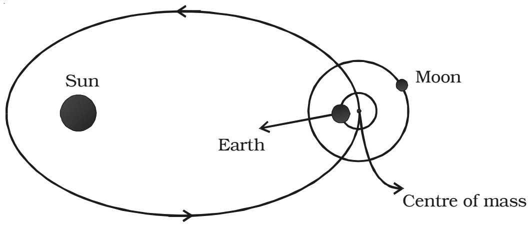


Fig 3.2 Centre of mass of Earth – moon system

For the system consisting of the Earth and the moon, their mutual gravitational attractions are the internal forces in the system and Sun's attraction on both the Earth and moon are the external forces acting on the centre of mass of the system.

3.1.3 Centre of gravity

A body may be considered to be made up of an indefinitely large number of particles, each of which is attracted towards the centre of the Earth by the force of gravity. These forces constitute a system of like parallel forces. The resultant of these parallel forces known as the weight of the body always acts through a point, which is fixed relative to the body, whatever be the position of the body. This fixed point is called the centre of gravity of the body.

The centre of gravity of a body is the point at which the resultant of the weights of all the particles of the body acts, whatever may be the orientation or position of the body provided that its size and shape remain unaltered.

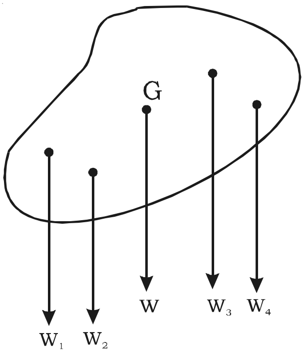


Fig . 3.3 Centre of gravity

In the Fig. 3.3, W_1, W_2, W_3..... are the weights of the first, second, third, ... particles in the body respectively. If W is the resultant weight of all the particles then the point at which W acts is known as the centre of gravity. The total weight of the body may be supposed to act at its centre of gravity. Since the weights of the particles constituting a body are practically proportional to their masses when the body is outside the Earth and near its surface, the centre of mass of a body practically coincides with its centre of gravity.

3.1.4 Equilibrium of bodies and types of equilibrium

If a marble *M* is placed on a curved surface of a bowl *S*, it rolls down and settles in equilibrium at the lowest point A (Fig. 3.4 a). This equilibrium position corresponds to minimum potential energy. If the marble is disturbed and displaced to a point *B*, its energy increases When it is released, the marble rolls back to *A*. Thus the marble at the position *A* is said to be in *stable equilibrium.*

Suppose now that the bowl *S* is inverted and the marble is placed at its top point, at *A* (Fig. 3.4b). If the marble is displaced slightly to the point *C*, its potential energy is lowered and tends to move further away from the equilibrium position to one of lowest energy. Thus the marble is said to be in *unstable equilibrium.*

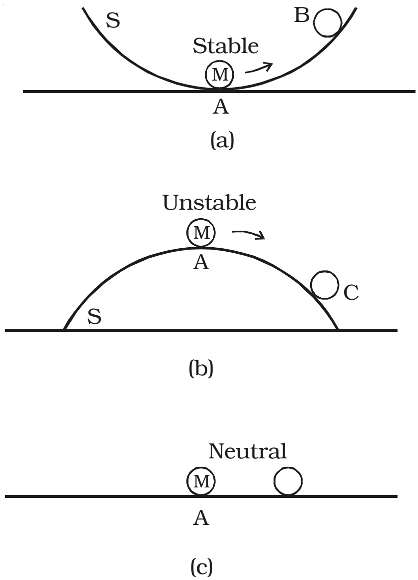


Fig.3.4 Equilibrium of rigid bodies

Suppose now that the marble is placed on a plane surface (Fig. 3.4c). If it is displaced slightly, its potential energy does not change. Here the marble is said to be in *neutral equilibrium.*

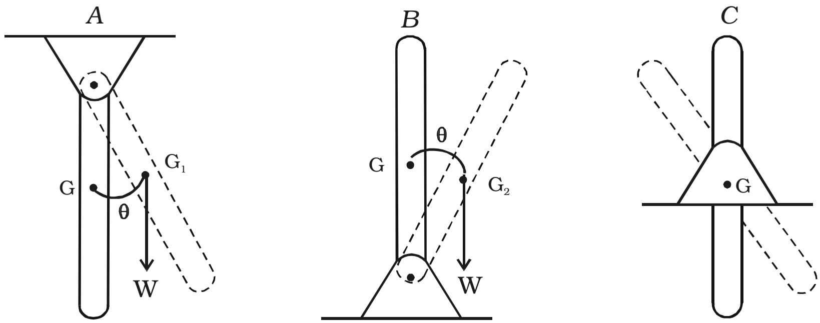
Equilibrium is thus stable, unstable or neutral according to whether the potential energy is minimum, maximum or constant.

We may also characterize the stability of a mechanical system by noting that when the system is disturbed from its position of equilibrium, the forces acting on the system may

(i) tend to bring back to its original position if potential energy is a *minimum,* corresponding to *stable equilibrium.*

(ii) tend to move it farther away if potential energy is *maximum,* corresponding *unstable equilibrium.*

(iii) tend to move either way if potential energy is a *constant* corresponding to *neutral equilibrium*



(a) Stable equilibrium (b) Unstable equilibrium (c) Neutral equilibrium

Fig 3.5 Types of equilibrium

Consider three uniform bars shown in Fig. 3.5 a,b,c. Suppose each bar is slightly displaced from its position of equilibrium and then released. For bar *A*, fixed at its top end, its centre of gravity G rises to G_1 on being displaced, then the bar returns back to its original position on being released, so that the equilibrium is stable.

For bar *B*, whose fixed end is at its bottom, its centre of gravity G is lowered to G_2 on being displaced, then the bar *B* will keep moving away from its original position on being released, and the equilibrium is said to be unstable.

For bar C, whose fixed point is about its centre of gravity, the centre of gravity remains at the same height on being displaced, the bar will remain in its new position, on being released, and the equilibrium is said to be neutral.

3.2 Rotational motion of rigid bodies

3.2.1 Rigid body

A rigid body is defined as that body which does not undergo any change in shape or volume when external forces are applied on it. When forces are applied on a rigid body, the distance between any two particles of the body will remain unchanged, however, large the forces may be.

Actually, no body is perfectly rigid. Every body can be deformed more or less by the application of the external force. The solids, in which the changes produced by external forces are negligibly small, are usually considered as rigid body.

3.2.2 Rotational motion

When a body rotates about a fixed axis, its motion is known as rotatory motion. *A rigid body is said to have pure rotational motion, if every particle of the body moves in a circle, the centre of which lies on a straight line called the axis of rotation* (Fig. 3.6). The axis of rotation may lie inside the body or even outside the body. The particles lying on the axis of rotation remains stationary.

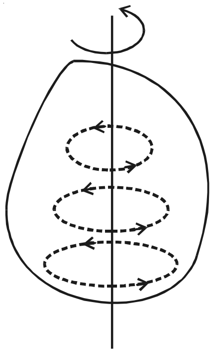

Axis of rotation

Fig 3.6 Rotational motion

The position of particles moving in a circular path is conveniently described in terms of a radius vector r and its angular displacement θ. Let us consider a rigid body that rotates about a fixed axis XOX′ passing through O and perpendicular to the plane of the paper as shown in Fig 3.7. Let the body rotate from the position A to the position B. The different particles at P_1, P_2, P_3. .... in the rigid body covers unequal distances P_1P_1', P_2P_2', P_3P_3'.... in the same interval of time. Thus their linear velocities are different. But in the same time interval, they all rotate through the same angle θ and hence the angular velocity is the same for the all the particles of the rigid body. Thus, *in the case of rotational motion, different constituent particles have different linear velocities but all of them have the same angular velocity.*

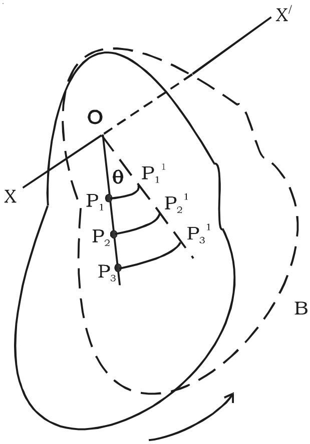


Fig 3.7 Rotational motion of a rigid body

3.2.3 Equations of rotational motion

As in linear motion, for a body having uniform angular acceleration, we shall derive the equations of motion.

Let us consider a particle start rotating with angular velocity ω_0 and angular acceleration α. At any instant t, let ω be the angular velocity of the particle and θ be the angular displacement produced by the particle.

Therefore change in angular velocity in time $t = \omega - \omega_0$

But, angular acceleration $= \dfrac{\textit{change in angular velocity}}{\textit{time taken}}$

(i.e) $\alpha = \frac{\omega - \omega_o}{t}$ ...(1)

$\omega = \omega_o + \alpha t$ ...(2)

The average angular velocity $= \left(\frac{\omega + \omega_o}{2}\right)$

The total angular displacement

= average angular velocity × time taken

(i.e) $\theta = \left(\frac{\omega + \omega_o}{2}\right) t$ ...(3)

Substituting ω from equation (2), $\theta = \left(\frac{\omega_o + \alpha t + \omega_o}{2}\right) t$

$\theta = \omega_o t + \frac{1}{2}\alpha t^2$ ...(4)

From equation (1), $t = \left(\frac{\omega - \omega_o}{\alpha}\right)$ ...(5)

using equation (5) in (3),

$$\theta = \left(\frac{\omega + \omega_o}{2}\right)\left(\frac{\omega - \omega_o}{\alpha}\right) = \frac{(\omega^2 - \omega_o^2)}{2\alpha}$$

$2\alpha\,\theta = \omega^2 - \omega_0^2$ or $\omega^2 = \omega_0^2 + 2\alpha\,\theta$ ...(6)

Equations (2), (4) and (6) are the equations of rotational motion.

3.3 Moment of inertia and its physical significance

According to Newton's first law of motion, a body must continue in its state of rest or of uniform motion unless it is compelled by some external agency called force. The inability of a material body to change its state of rest or of uniform motion by itself is called inertia. Inertia is the fundamental property of the matter. For a given force, the greater the mass, the higher will be the opposition for motion, or larger the inertia. Thus, in translatory motion, the mass of the body measures the coefficient of inertia.

Similarly, in rotational motion also, a body, which is free to rotate about a given axis, opposes any change desired to be produced in its state. The measure of opposition will depend on the mass of the body and the distribution of mass about the axis of rotation. The coefficient of inertia in rotational motion is called the moment of inertia of the body about the given axis.

Moment of inertia plays the same role in rotational motion as that of mass in translatory motion. Also, to bring about a change in the state of rotation, torque has to be applied.

3.3.1 Rotational kinetic energy and moment of inertia of a rigid body

Consider a rigid body rotating with angular velocity ω about an axis XOX′. Consider the particles of masses m_1, m_2, m_3... situated at distances r_1, r_2, r_3... respectively from the axis of rotation. The angular velocity of all the particles is same but the particles rotate with different linear velocities. Let the linear velocities of the particles be v_1, v_2, v_3 ... respectively.

Kinetic energy of the first particle = $\frac{1}{2} m_1 v_1^2$

But $v_1 = r_1\omega$

∴ Kinetic energy of the first particle

$$= \frac{1}{2} m_1 (r_1\omega)^2 = \frac{1}{2} m_1 r_1^2 \omega^2$$

Similarly,

Kinetic energy of second particle

$$= \frac{1}{2} m_2 r_2^2 \omega^2$$

Kinetic energy of third particle

$$= \frac{1}{2} m_3 r_3^2 \omega^2$$ and so on.

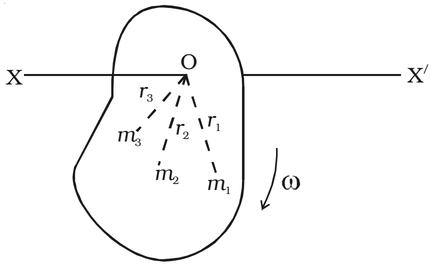


Fig. 3.8 Rotational kinetic energy and moment of inertia

The kinetic energy of the rotating rigid body is equal to the sum of the kinetic energies of all the particles.

∴ Rotational kinetic energy

$$= \frac{1}{2} (m_1 r_1^2 \omega^2 + m_2 r_2^2 \omega^2 + m_3 r_3^2 \omega^2 + ... + m_n r_n^2 w^2)$$

$$= \frac{1}{2} \omega^2 (m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + ... + m_n r_n^2)$$

(i.e) $$E_R = \frac{1}{2}\omega^2\left(\sum_{i=1}^{n} m_i r_i^2\right) \quad ...(1)$$

In translatory motion, kinetic energy $= \frac{1}{2}mv^2$

Comparing with the above equation, the inertial role is played by the term $\sum_{i=1}^{n} m_i r_i^2$. This is known as moment of inertia of the rotating rigid body about the axis of rotation. Therefore the moment of inertia is

I = mass × (distance)^2

Kinetic energy of rotation $= \frac{1}{2}\omega^2 I$

When ω = 1 rad s^{-1}, rotational kinetic energy

$$= E_R = \frac{1}{2}(1)^2 I \quad (or) \quad I = 2E_R$$

It shows that moment of inertia of a body is equal to twice the kinetic energy of a rotating body whose angular velocity is one radian per second.

The unit for moment of inertia is *kg m*^{2} and the dimensional formula is ML^2.

3.3.2 Radius of gyration

The moment of inertia of the rotating rigid body is,

$$I = \sum_{i=1}^{n} m_i r_i^2 = m_1 r_1^2 + m_2 r_2^2 + ...m_n r_n^2$$

If the particles of the rigid body are having same mass, then

$$m_1 = m_2 = m_3 = ... = m \text{ (say)}$$

∴ The above equation becomes,

$$I = mr_1^2 + mr_2^2 + mr_3^2 + ... + mr_n^2$$

$$= m(r_1^2 + r_2^2 + r_3^2 + ... + r_n^2)$$

$$I = nm\left[\frac{r_1^2 + r_2^2 + r_3^2 ..... + r_n^2}{n}\right]$$

where *n* is the number of particles in the rigid body.

$$\therefore \quad I = MK^2 \qquad \ldots (2)$$

where $M = nm$, total mass of the body and $K^2 = \dfrac{r_1^2 + r_2^2 + r_3^2 \ldots\ldots + r_n^2}{n}$

Here $K = \sqrt{\dfrac{r_1^2 + r_2^2 + r_3^2 \ldots\ldots + r_n^2}{n}}$ is called as the radius of gyration of the rigid body about the axis of rotation.

The radius of gyration is equal to the root mean square distances of the particles from the axis of rotation of the body.

The radius of gyration can also be defined as the perpendicular distance between the axis of rotation and the point where the whole weight of the body is to be concentrated.

Also from the equation (2) $K^2 = \frac{I}{M}$ (or) $K = \sqrt{\frac{I}{M}}$

3.3.3 *Theorems of moment of inertia*

(i) Parallel axes theorem

Statement

The moment of inertia of a body about any axis is equal to the sum of its moment of inertia about a parallel axis through its centre of gravity and the product of the mass of the body and the square of the distance between the two axes.

Proof

Let us consider a body having its centre of gravity at G as shown in Fig. 3.9. The axis XX′ passes through the centre of gravity and is perpendicular to the plane of the body. The axis X_1X_1' passes through the point O and is parallel to the axis XX′. The distance between the two parallel axes is x.

Let the body be divided into large number of particles each of mass m. For a particle P at a distance r from O, its moment of inertia about the axis X_1OX_1' is equal to $m r^2$.

The moment of inertia of the whole body about the axis X_1X_1' is given by,

$$I_0 = \Sigma\, mr^2 \qquad \ldots(1)$$

From the point P, drop a perpendicular PA to the extended OG and join PG.

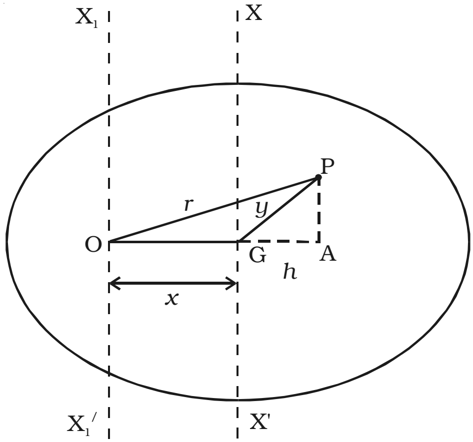


Fig .3.9 Parallel axes theorem

In the ΔOPA,

$OP^2 = OA^2 + AP^2$

$r^2 = (x + h)^2 + AP^2$

$r^2 = x^2 + 2xh + h^2 + AP^2$ ...(2)

But from Δ GPA,

$GP^2 = GA^2 + AP^2$

$y^2 = h^2 + AP^2$ ...(3)

Substituting equation (3) in (2),

$r^2 = x^2 + 2xh + y^2$ ...(4)

Substituting equation (4) in (1),

$I_o = \Sigma m (x^2 + 2xh + y^2)$

$= \Sigma mx^2 + \Sigma 2mxh + \Sigma my^2$

$= Mx^2 + My^2 + 2x\Sigma mh$ ...(5)

Here $My^2 = I_G$ is the moment of inertia of the body about the line passing through the centre of gravity. The sum of the turning moments of all the particles about the centre of gravity is zero, since the body is balanced about the centre of gravity G.

$\Sigma (mg) (h) = 0$ (or) $\Sigma mh = 0$ [since g is a constant] ...(6)

$\therefore$ equation (5) becomes, $I_0 = Mx^2 + I_G$ ...(7)

Thus the parallel axes theorem is proved.

(ii) Perpendicular axes theorem

Statement

The moment of inertia of a plane laminar body about an axis perpendicular to the plane is equal to the sum of the moments of inertia about two mutually perpendicular axes in the plane of the lamina such that the three mutually perpendicular axes have a common point of intersection.

Proof

Consider a plane lamina having the axes *OX* and *OY* in the plane of the lamina as shown Fig. 3.10. The axis *OZ* passes through *O* and is perpendicular to the plane of the lamina. Let the lamina be divided into a large number of particles, each of mass *m*. A particle at *P* at a distance *r* from *O* has coordinates (x,y).

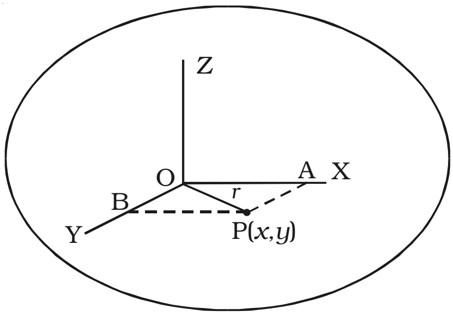


Fig 3.10 Perpendicular axes theorem

$\therefore r^2 = x^2+y^2$ ...(1)

The moment of inertia of the particle *P* about the axis $OZ = m\,r^2$.

The moment of inertia of the whole lamina about the axis *OZ* is

$I_z = \Sigma mr^2$ ...(2)

The moment of inertia of the whole lamina about the axis *OX* is

$I_x = \Sigma\, my^2$ ...(3)

Similarly, $I_y = \Sigma\, mx^2$ ...(4)

From eqn. (2), $I_z = \Sigma mr^2 = \Sigma m(x^2+y^2)$

$I_z = \Sigma mx^2 + \Sigma my^2 = I_y + I_x$

$\therefore I_z = I_x + I_y$

which proves the perpendicular axes theorem.

Table 3.1 Moment of Inertia of different bodies

(Proof is given in the annexure)

Body	Axis of Rotation	Moment of Inertia	
Thin Uniform Rod	Axis passing through its centre of gravity and perpendicular to its length	$\frac{Ml^2}{12}$	M - mass l - length
	Axis passing through the end and perpendicular to its length.	$\frac{Ml^2}{3}$	M - mass l - length
Thin Circular Ring	Axis passing through its centre and perpendicular to its plane.	MR^2	M - mass R - radius
	Axis passing through its diameter	$\frac{1}{2}MR^2$	M - mass R - radius
	Axis passing through a tangent	$\frac{3}{2}MR^2$	M - mass R - radius
Circular Disc	Axis passing through its centre and perpendicular to its plane.	$\frac{1}{2}MR^2$	M - mass R - radius
	Axis passing through its diameter	$\frac{1}{4}MR^2$	M - mass R - radius
	Axis passing through a tangent	$\frac{5}{4}MR^2$	M - mass R - radius
Solid Sphere	Axis passing through its diameter	$\frac{2}{5}MR^2$	M - mass R - radius
	Axis passing through a tangent	$\frac{7}{5}MR^2$	M - mass R - radius
Solid Cylinder	Its own axis	$\frac{1}{2}MR^2$	M - mass R - radius
	Axis passing through its centre and perpedicular to its length	$M\left(\frac{R^2}{4}+\frac{l^2}{12}\right)$	M - mass R - radius l - length

3.4 Moment of a force

A force can rotate a nut when applied by a wrench or it can open a door while the door rotates on its hinges (i.e) in addition to the tendency to move a body in the direction of the application of a force, a force also tends to rotate the body about any axis which does not intersect the line of action of the force and also not parallel to it. This tendency of rotation is called turning effect of a force or moment of the force about the given axis. *The magnitude of the moment of force F about a point is defined as the product of the magnitude of force and the perpendicular distance of the point from the line of action of the force.*

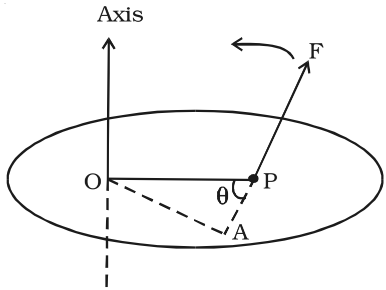


Fig 3.11 Moment of a force

Let us consider a force *F* acting at the point P on the body as shown in Fig. 3.11. Then, the moment of the force F about the point O = Magnitude of the force × perpendicular distance between the direction of the force and the point about which moment is to be determined = F × OA.

If the force acting on a body rotates the body in anticlockwise direction with respect to O then the moment is called anticlockwise moment. On the other hand, if the force rotates the body in clockwise direction then the moment is said to be clockwise moment. The unit of moment of the force is N m and its dimensional formula is $M L^2 T^{-2}$.

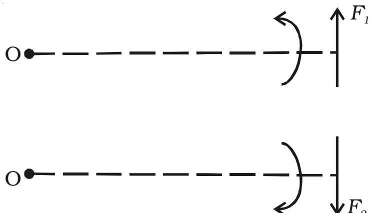


Fig 3.12 Clockwise and anticlockwise moments

As a matter of convention,an anticlockwise moment is taken as positive and a clockwise moment as negative. While adding moments, the direction of each moment should be taken into account.

In terms of vector product, the moment of a force is expressed as,

$$\vec{m} = \vec{r} \times \vec{F}$$

where $\vec{r}$ is the position vector with respect to O. The direction of $\vec{m}$ is perpendicular to the plane containing $\vec{r}$ and $\vec{F}$.

3.5 Couple and moment of the couple (Torque)

There are many examples in practice where two forces, acting together, exert a moment, or turning effect on some object. As a very simple case, suppose two strings are tied to a wheel at the points *X* and *Y*, and *two equal and opposite forces, F*, are exerted tangentially to the wheels (Fig. 3.13). If the wheel is pivoted at its centre *O* it begins to rotate about *O* in an anticlockwise direction.

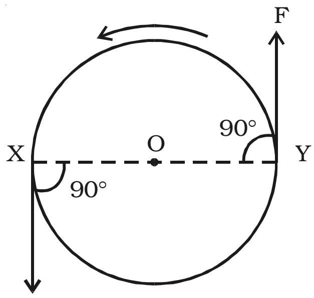


Fig. 3.13 Couple

Two equal and opposite forces whose lines of action do not coincide are said to constitute a couple in mechanics. The two forces always have a *turning effect*, or *moment*, called a *torque*. The perpendicular distance between the lines of action of two forces, which constitute the couple, is called the *arm of the couple.*

The product of the forces forming the couple and the arm of the couple is called the moment of the couple or torque.

Torque = one of the forces × perpendicular distance between the forces

The torque in rotational motion plays the same role as the force in translational motion. A quantity that is a measure of this rotational effect produced by the force is called torque.

In vector notation, $\vec{\tau} = \vec{r} \times \vec{F}$

The torque is maximum when θ = 90° (i.e) when the applied force is at right angles to $\vec{r}$.

Examples of couple are

1. Forces applied to the handle of a screw press,
2. Opening or closing a water tap.
3. Turning the cap of a pen.
4. Steering a car.

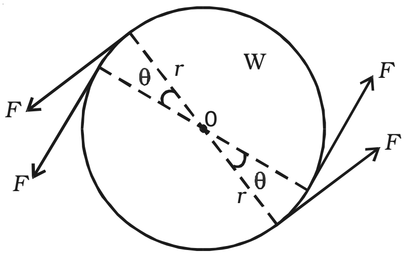


Fig.3.14 Work done by a couple

Work done by a couple

Suppose two equal and opposite forces *F* act tangentially to a wheel *W*, and rotate it through an angle θ (Fig. 3.14).

Then the work done by each force = Force × distance = $F \times r\,\theta$

(since $r\ \theta$ is the distance moved by a point on the rim)

Total work done $W = F\,r\,\theta + F\,r\,\theta = 2F\,r\,\theta$

but torque $\tau = F \times 2r = 2F\,r$

$\therefore$ work done by the couple, $W = \tau\,\theta$

3.6 *Angular momentum of a particle*

The angular momentum in a rotational motion is similar to the linear momentum in translatory motion. The linear momentum of a particle moving along a straight line is the product of its mass and linear velocity (i.e) $p = mv$. *The angular momentum of a particle is defined as the moment of linear momentum of the particle.*

Let us consider a particle of mass m moving in the *XY* plane with a velocity v and linear momentum $\vec{p} = m\vec{v}$ at a distance r from the origin (Fig. 3.15).

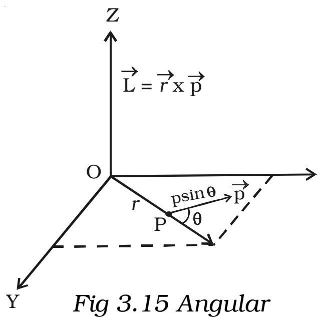


Fig 3.15 Angular momentum of a particle

The angular momentum L of the particle about an axis passing through O perpendicular to XY plane is defined as the cross product of $\vec{r}$ and $\vec{p}$.

(i.e) $\vec{L} = \vec{r} \times \vec{p}$

Its magnitude is given by $L = r\,p\,\sin\theta$

where θ is the angle between $\vec{r}$ and $\vec{p}$ and L is along a direction perpendicular to the plane containing $\vec{r}$ and $\vec{p}$.

The unit of angular momentum is kg m^2 s^{-1} and its dimensional formula is, $M\,L^2\,T^{-1}$.

3.6.1 *Angular momentum of a rigid body*

Let us consider a system of n particles of masses m_1, m_2 ..... m_n situated at distances r_1, r_2, ...r_n respectively from the axis of rotation (Fig. 3.16). Let v_1, v_2, v_3 ..... be the linear velocities of the particles respectively, then linear momentum of first particle $= m_1 v_1$.

Since $v_1 = r_1\omega$ the linear momentum of first particle = $m_1(r_1\,\omega)$

The moment of linear momentum of first particle

= linear momentum × perpendicular distance

= $(m_1 r_1 \omega) \times r_1$

angular momentum of first particle = $m_1 r_1^2 \omega$

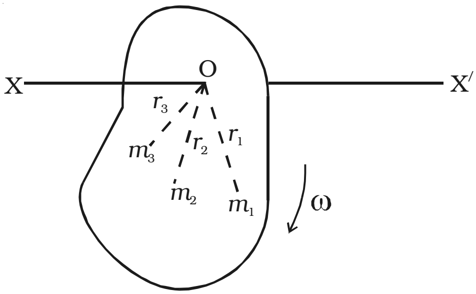


Fig 3.16 Angular momentum of a rigid body

Similarly,

angular momentum of second particle = $m_2 r_2^2 \omega$

angular momentum of third particle = $m_3 r_3^2 \omega$ and so on.

The sum of the moment of the linear momenta of all the particles of a rotating rigid body taken together about the axis of rotation is known as angular momentum of the rigid body.

∴ Angular momentum of the rotating rigid body = sum of the angular momenta of all the particles.

(i.e) $L = m_1 r_1^2 \omega + m_2 r_2^2 \omega + m_3 r_3^2 \omega \;..... + m_n r_n^2 \omega$

$L = \omega\,[m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + ...m_n r_n^2]$

$$= \omega \left[\sum_{i=1}^{n} m_i r_i^2 \right]$$

$\therefore \quad L = \omega I$

where $I = \sum_{i=1}^{n} m_i r_i^2$ = moment of inertia of the rotating rigid body about the axis of rotation.

3.7 Relation between torque and angular acceleration

Let us consider a rigid body rotating about a fixed axis X0X′ with angular velocity ω (Fig. 3.17).

The force acting on a particle of mass m_1 situated at A, at a distance r_1, from the axis of rotation = mass × acceleration

$$= m_1 \times \frac{d}{dt}(r_1\omega)$$

$$= m_1 r_1 \frac{d\omega}{dt}$$

$$= m_1 r_1 \frac{d^2\theta}{dt^2}$$

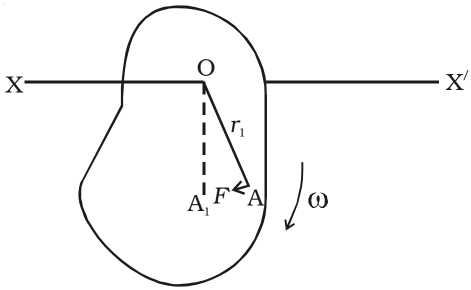


Fig 3.17 Relation between torque and angular acceleration

The moment of this force about the axis of rotation

= Force × perpendicular distance

$$= m_1 r_1 \frac{d^2\theta}{dt^2} \times r_1$$

Therefore, the total moment of all the forces acting on all the particles

$$= m_1 r_1^2 \frac{d^2\theta}{dt^2} + m_2 r_2^2 \frac{d^2\theta}{dt^2} + \ldots$$

(i.e) torque $= \sum_{i=1}^{n} m_i r_i^2 \times \frac{d^2\theta}{dt^2}$

or $\tau = I\alpha$

where $\sum_{i=1}^{n} m_i r_i^2$ = moment of inertia I of the rigid body and $\alpha = \frac{d^2\theta}{dt^2}$ angular acceleration.

3.7.1 Relation between torque and angular momentum

The angular momentum of a rotating rigid body is, $L = I\,\omega$

Differentiating the above equation with respect to time,

$$\frac{dL}{dt} = I\left(\frac{d\omega}{dt}\right) = I\alpha$$

where $\alpha = \frac{d\omega}{dt}$ angular acceleration of the body.

But torque $\tau = I\alpha$

Therefore, torque $\tau = \frac{dL}{dt}$

Thus the rate of change of angular momentum of a body is equal to the external torque acting upon the body.

3.8 Conservation of angular momentum

The angular momentum of a rotating rigid body is, $L = I\,\omega$

The torque acting on a rigid body is, $\tau = \frac{dL}{dt}$

When no external torque acts on the system, $\tau = \frac{dL}{dt} = 0$

(i.e) $L = I\omega$ = constant

Total angular momentum of the body = constant

(i.e.) when no external torque acts on the body, the net angular momentum of a rotating rigid body remains constant. This is known as law of conservation of angular momentum.

Illustration of conservation of angular momentum

From the law of conservation of angular momentum, $I\omega$ = constant

(ie) $\omega \propto \frac{1}{I}$, the angular velocity of rotation is inversely proportional to the moment of inertia of the system.

Following are the examples for law of conservation of angular momentum.

1. A diver jumping from springboard sometimes exhibits somersaults in air before reaching the water surface, because the diver curls his body to decrease the moment of inertia and increase angular velocity. When he

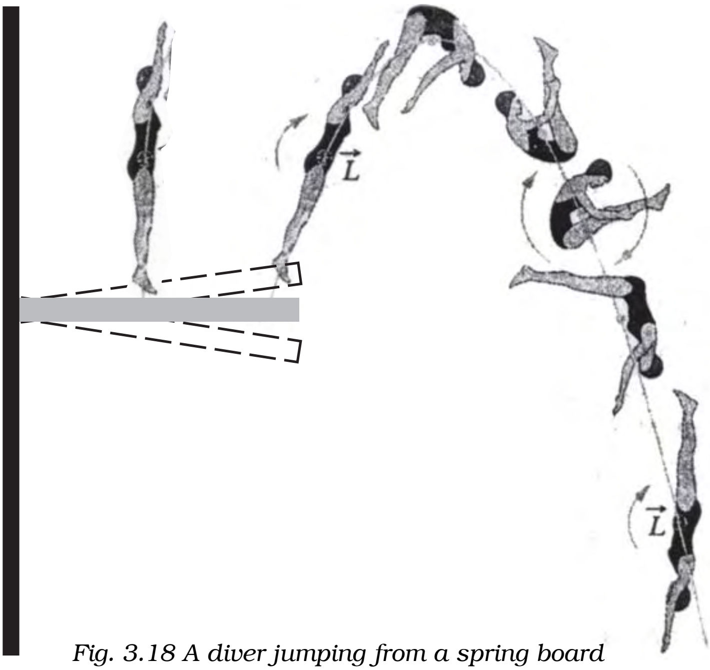


Fig. 3.18 A diver jumping from a spring board

is about to reach the water surface, he again outstretches his limbs. This again increases moment of inertia and decreases the angular velocity. Hence, the diver enters the water surface with a gentle speed.

2. A ballet dancer can increase her angular velocity by folding her arms, as this decreases the moment of inertia.

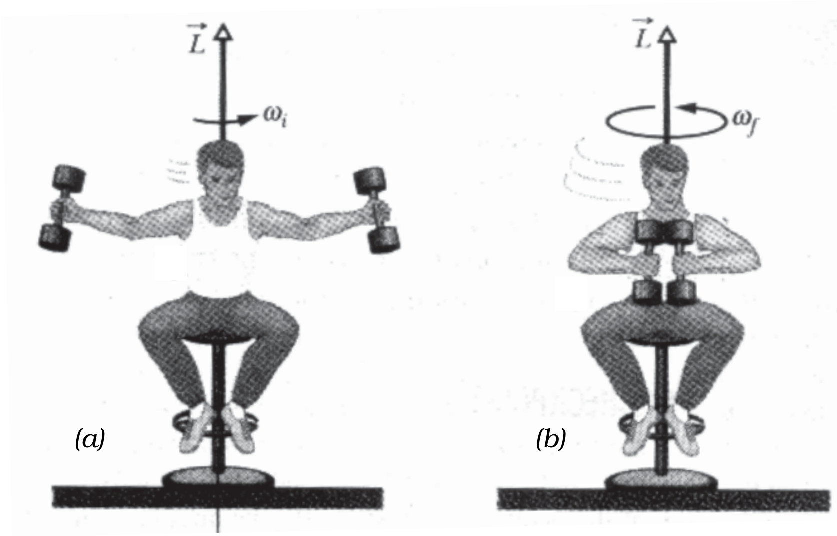


Fig 3.19 A person rotating on a turn table

3. Fig. 3.19a shows a person sitting on a turntable holding a pair of heavy dumbbells one in each hand with arms outstretched. The table is rotating with a certain angular velocity. The person suddenly pushes the weight towards his chest as shown Fig. 3.19b, the speed of rotation is found to increase considerably.

4.The angular velocity of a planet in its orbit round the sun increases when it is nearer to the Sun, as the moment of inertia of the planet about the Sun decreases.

Solved Problems

3.1 A system consisting of two masses connected by a massless rod lies along the X-axis. A 0.4 kg mass is at a distance $x = 2$ m while a 0.6 kg mass is at $x = 7$ m. Find the x coordinate of the centre of mass.

Data : $m_1 = 0.4\ kg$; $m_2 = 0.6\ kg$; $x_1 = 2\ m$; $x_2 = 7\ m$; $x = ?$

Solution : $x = \dfrac{m_1x_1 + m_2x_2}{m_1 + m_2} = \dfrac{(0.4 \times 2) + (0.6 \times 7)}{(0.4 + 0.6)} = 5\ m$

3.2 Locate the centre of mass of a system of bodies of masses $m_1 = 1$ kg, $m_2 = 2$ kg and $m_3 = 3$ kg situated at the corners of an equilateral triangle of side 1 m.

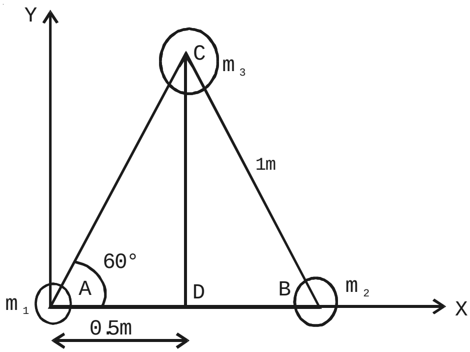
Data : $m_1 = 1\ kg$; $m_2 = 2\ kg$; $m_3 = 3\ kg$;

The coordinates of A = (0,0)

The coordinates of B =(1,0)

Centre of mass of the system =?

Solution : *Consider an equilateral triangle of side 1m as shown in Fig. Take X and Y axes as shown in figure.*



To find the coordinate of C:

For an equilateral triangle , $\angle CAB = 60°$

Consider the triangle ADC,

$\sin\theta = \dfrac{CD}{CA}$ (or) $CD = (CA)\ \sin\theta = 1 \times \sin 60° = \dfrac{\sqrt{3}}{2}$

Therefore from the figure, the coordinate of C are, $\left(0.5, \dfrac{\sqrt{3}}{2} \right)$

$$x = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3}$$

$$x=\frac{(1\times0)+(2\times1)+(3\times0.5)}{(1+2+3)}=\frac{3.5}{6}m$$

$$y=\frac{m_1y_1+m_2y_2+m_3y_3}{m_1+m_2+m_3}$$

$$y=\frac{(1\times0)+(2\times0)+\left(3\times\frac{\sqrt{3}}{2}\right)}{6}=\frac{\sqrt{3}}{4}m$$

3.3 A circular disc of mass m and radius r is set rolling on a table. If ω is its angular velocity, show that its total energy $E=\frac{3}{4}mr^2\omega^2$.

Solution : *The total energy of the disc = Rotational KE + linear KE*

$$\therefore\ E=\frac{1}{2}I\omega^2+\frac{1}{2}mv^2 \qquad ...(1)$$

But $$I=\frac{1}{2}mr^2 \text{ and } v=r\omega \qquad ...(2)$$

Substituting eqn. (2) in eqn. (1),

$$E=\frac{1}{2}\times\frac{1}{2}(mr^2)(\omega^2)+\frac{1}{2}m(r\omega)^2=\frac{1}{4}mr^2\omega^2+\frac{1}{2}mr^2\omega^2$$

$$=\frac{3}{4}mr^2\omega^2$$

3.4 A thin metal ring of diameter 0.6m and mass 1kg starts from rest and rolls down on an inclined plane. Its linear velocity on reaching the foot of the plane is 5 m s^{-1}, calculate (i) the moment of inertia of the ring and (ii) the kinetic energy of rotation at that instant.

Data : $R = 0.3$ m ; $M = 1$ kg ; $v = 5$ m s^{-1} ; I = ? K.E. = ?

Solution : $I = MR^2 = 1\times(0.3)^2 = 0.09$ kg m^2

$$K.E.=\frac{1}{2}I\omega^2$$

$$v=r\omega\ ;\quad \therefore\ \omega=\frac{v}{r}\ ;\quad K.E.=\frac{1}{2}\times0.09\times\left(\frac{5}{0.3}\right)^2=12.5\ J$$

3.5 A solid cylinder of mass 200 kg rotates about its axis with angular speed 100 s^{-1}. The radius of the cylinder is 0.25 m. What is the kinetic energy associated with the rotation of the cylinder? What is the magnitude of the angular momentum of the cylinder about its axis?

Data : $M = 200\ kg\ ;\ \omega = 100\ s^{-1}\ ;\ R = 0.25$ *metre* ; $E_R = ?\ ;\ L = ?$

Solution : $I = \dfrac{MR^2}{2} = \dfrac{200 \times (0.25)^2}{2} = 6.25\ kg\ m^2$

$$K.E. = \frac{1}{2} I\,\omega^2$$

$$= \frac{1}{2} \times 6.25 \times (100)^2$$

$$E_R = 3.125 \times 10^4\ J$$

$$L = I\omega = 6.25 \times 100 = 625\ kg\ m^2\ s^{-1}$$

3.6 Calculate the radius of gyration of a rod of mass 100 g and length 100 cm about an axis passing through its centre of gravity and perpendicular to its length.

Data : $M = 100\ g = 0.1\ kg \quad l = 100\ cm = 1\ m$

$K = ?$

Solution : *The moment of inertia of the rod about an axis passing through its centre of gravity and perpendicular to the length* = I =

$$MK^2 = \frac{ML^2}{12} \text{ (or) } K^2 = \frac{L^2}{12} \text{ (or) } K = \frac{L}{\sqrt{12}} = \frac{1}{\sqrt{12}} = 0.2886\ m.$$

3.7 A circular disc of mass 100 g and radius 10 cm is making 2 revolutions per second about an axis passing through its centre and perpendicular to its plane. Calculate its kinetic energy.

Data : $M = 100\ g = 0.1\ kg\ ;\ R = 10\ cm = 0.1\ m\ ;\ n = 2$

Solution : ω = *angular velocity* = $2\pi n = 2\pi \times 2 = 4\pi$ rad / s

Kinetic energy of rotation = $\dfrac{1}{2} I\omega^2$

$$= \frac{1}{2} \times \frac{1}{2} \times MR^2\,\omega^2 = \frac{1}{2} \times \frac{1}{2}\ (0.1) \times (0.1)^2 \times (4\pi)^2$$

$$= 3.947 \times 10^{-2}\ J$$

3.8 Starting from rest, the flywheel of a motor attains an angular velocity 100 rad/s from rest in 10 s. Calculate (i) angular acceleration and (ii) angular displacement in 10 seconds.

Data : $\omega_o = 0$; $\omega = 100\ rad\ s^{-1}$ $t = 10\ s$ $\alpha = ?$

Solution : *From equations of rotational dynamics,*

$$\omega = \omega_0 + at$$

$$(or)\ \alpha = \frac{\omega - \omega_o}{t} = \frac{100-0}{10} = 10\ \text{rad}\ s^{-2}$$

Angular displacement $\theta = \omega_o t + \frac{1}{2}\alpha t^2$

$$= 0 + \frac{1}{2} \times 10 \times 10^2 = 500\ rad$$

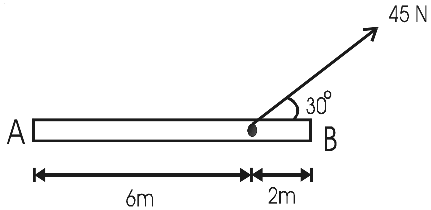
3.9 A disc of radius 5 cm has moment of inertia of 0.02 kg m^2.A force of 20 N is applied tangentially to the surface of the disc. Find the angular acceleration produced.

Data : $I = 0.02\ kg\ m^2$; $r = 5\ cm = 5 \times 10^{-2}\ m$; $F = 20\ N$; $\tau = ?$

Solution : Torque $= \tau = F \times 2r = 20 \times 2 \times 5 \times 10^{-2} = 2\ N\ m$

angular acceleration $= \alpha = \frac{\tau}{I} = \frac{2}{0.02} = 100\ rad\ /s^2$

3.10 From the figure, find the moment of the force 45 N about A?

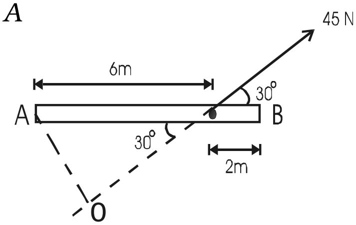


Data : *Force F = 45 N ; Moment of the force about A = ?*

Solution : *Moment of the force about A*

= *Force × perpendicular distance* = $F \times AO$

$= 45 \times 6 \sin 30 = 135\ N\ m$



4. Gravitation and Space Science

We have briefly discussed the kinematics of a freely falling body under the gravity of the Earth in earlier units. The fundamental forces of nature are gravitational, electromagnetic and nuclear forces. The gravitational force is the weakest among them. But this force plays an important role in the birth of a star, controlling the orbits of planets and evolution of the whole universe.

Before the seventeenth century, scientists believed that objects fell on the Earth due to their inherent property of matter. Galileo made a systematic study of freely falling bodies.

4.1 Newton's law of gravitation

The motion of the planets, the moon and the Sun was the interesting subject among the students of Trinity college at Cambridge in England. Isaac Newton was also one among these students. In 1665, the college was closed for an indefinite period due to plague. Newton, who was then 23 years old, went home to Lincolnshire. He continued to think about the motion of planets and the moon. One day Newton sat under an apple tree and had tea with his friends. He saw an apple falling to ground. This incident made him to think about falling bodies. He concluded that the same force of gravitation which attracts the apple to the Earth might also be responsible for attracting the moon and keeping it in its orbit. The centripetal acceleration of the moon in its orbit and the downward acceleration of a body falling on the Earth might have the same origin. Newton calculated the centripetal acceleration by assuming moon's orbit (Fig. 4.1) to be circular.

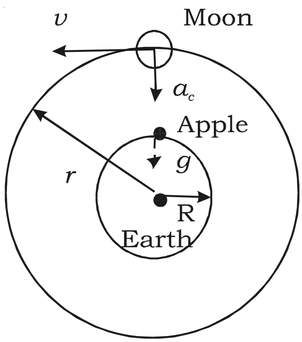


Fig. 4.1 Acceleration of moon

Acceleration due to gravity on the Earth's surface, g = 9.8 m s^{-2}

Centripetal acceleration on the moon, $a_c = \dfrac{v^2}{r}$

where r is the radius of the orbit of the moon (3.84×10^8 m) and v is the speed of the moon.

Time period of revolution of the moon around the Earth,

T = 27.3 days.

The speed of the moon in its orbit, $v = \frac{2\pi r}{T}$

$$v = \frac{2\pi \times 3.84 \times 10^8}{27.3 \times 24 \times 60 \times 60} = 1.02 \times 10^3 \text{ m s}^{-1}$$

$$\therefore \text{ Centripetal acceleration, } \quad a_c = \frac{v^2}{r} = \frac{(1.02 \times 10^3)^2}{3.84 \times 10^8}$$

$$a_c = 2.7 \times 10^{-3} \text{ m s}^{-2}$$

Newton assumed that both the moon and the apple are accelerated towards the centre of the Earth. But their motions differ, because, the moon has a tangential velocity whereas the apple does not have.

Newton found that a_c was less than g and hence concluded that force produced due to gravitational attraction of the Earth decreases with increase in distance from the centre of the Earth. He assumed that this acceleration and therefore force was inversely proportional to the square of the distance from the centre of the Earth. He had found that the value of a_c was about 1/3600 of the value of g, since the radius of the lunar orbit r is nearly 60 times the radius of the Earth R.

The value of a_c was calculated as follows :

$$\frac{a_c}{g} = \frac{1/r^2}{1/R^2} = \left(\frac{R}{r}\right)^2 = \left(\frac{1}{60}\right)^2 = \frac{1}{3600}$$

$$\therefore \ a_c = \frac{g}{3600} = \frac{9.8}{3600} = 2.7 \times 10^{-3} \text{ m s}^{-2}$$

Newton suggested that gravitational force might vary inversely as the square of the distance between the bodies. He realised that this force of attraction was a case of universal attraction between any two bodies present anywhere in the universe and proposed universal gravitational law.

The law states that *every particle of matter in the universe attracts every other particle with a force which is directly proportional to the product* *of their masses and inversely proportional to the square of the distance between them.*

Consider two bodies of masses m_1 and m_2 with their centres separated by a distance r. The gravitational force between them is

$F \propto m_1 m_2$

$F \propto 1/r^2$

$\therefore F \propto \frac{m_1 m_2}{r^2}$

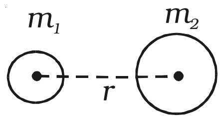


Fig. 4.2 Gravitational force

$F = G \frac{m_1 m_2}{r^2}$ where G is the universal gravitational constant.

If $m_1 = m_2 = 1$ kg and $r = 1$ m, then $F = G$.

Hence, the Gravitational constant 'G' is numerically equal to the gravitational force of attraction between two bodies of mass 1 kg each separated by a distance of 1 m. The value of G is 6.67×10^{-11} N m^2 kg^{-2} and its dimensional formula is $M^{-1} L^3 T^{-2}$.

4.1.1 *Special features of the law*

(i) The gravitational force between two bodies is an action and reaction pair.

(ii) The gravitational force is very small in the case of lighter bodies. It is appreciable in the case of massive bodies. The gravitational force between the Sun and the Earth is of the order of 10^{27} N.

4.2 Acceleration due to gravity

Galileo was the first to make a systematic study of the motion of a body under the gravity of the Earth. He dropped various objects from the leaning tower of Pisa and made analysis of their motion under gravity. He came to the conclusion that *"in the absence of air, all bodies will fall at the same rate"*. It is the air resistance that slows down a piece of paper or a parachute falling under gravity. If a heavy stone and a parachute are dropped where there is no air, both will fall together at the same rate.

Experiments showed that the velocity of a freely falling body under gravity increases at a constant rate. (i.e) with a constant acceleration. The acceleration produced in a body on account of the force of gravity is called *acceleration due to gravity*. It is denoted by *g*. At a given place, the value of *g* is the same for all bodies irrespective of their masses. It differs from place to place on the surface of the Earth. It also varies with altitude and depth.

The value of *g* at sea–level and at a latitude of 45° is taken as the standard (i.e) $g = 9.8 \text{ m s}^{-2}$

4.3 Acceleration due to gravity at the surface of the Earth

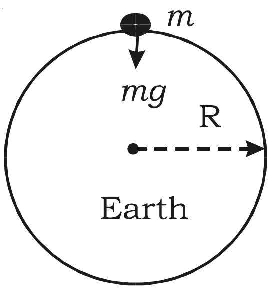


Fig. 4.3 Acceleration due to gravity

Consider a body of mass m on the surface of the Earth as shown in the Fig. 4.3. Its distance from the centre of the Earth is *R* (radius of the Earth).

The gravitational force experienced by the body is $F = \frac{GMm}{R^2}$ where *M* is the mass of the Earth.

From Newton's second law of motion,

Force $F = mg$.

Equating the above two forces, $\frac{GMm}{R^2} = mg$

$$\therefore g = \frac{GM}{R^2}$$

This equation shows that *g* is independent of the mass of the body *m*. But, it varies with the distance from the centre of the Earth. If the Earth is assumed to be a sphere of radius *R*, the value of *g* on the surface of the Earth is given by $g = \frac{GM}{R^2}$

4.3.1 Mass of the Earth

From the expression $g = \frac{GM}{R^2}$, the mass of the Earth can be calculated as follows :

$$M = \frac{gR^2}{G} = \frac{9.8 \times (6.38 \times 10^6)^2}{6.67 \times 10^{-11}} = 5.98 \times 10^{24} \text{ kg}$$

4.4 Variation of acceleration due to gravity

(i) Variation of g with altitude

Let P be a point on the surface of the Earth and Q be a point at an altitude h. Let the mass of the Earth be M and radius of the Earth be R. Consider the Earth as a spherical shaped body.

The acceleration due to gravity at P on the surface is

$$g = \frac{GM}{R^2} \quad \text{... (1)}$$

Let the body be placed at Q at a height h from the surface of the Earth. The acceleration due to gravity at Q is

$$g_h = \frac{GM}{(R+h)^2} \quad \text{... (2)}$$

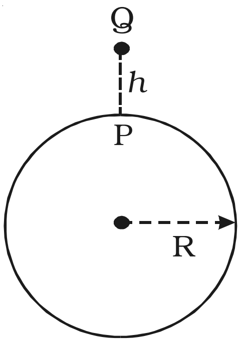


Fig. 4.4 Variation of g with altitude

dividing (2) by (1) $\frac{g_h}{g} = \frac{R^2}{(R+h)^2}$

By simplifying and expanding using binomial theorem, $g_h = g\left(1 - \frac{2h}{R}\right)$

The value of acceleration due to gravity decreases with increase in height above the surface of the Earth.

(ii) Variation of g with depth

Consider the Earth to be a homogeneous sphere with uniform density of radius R and mass M.

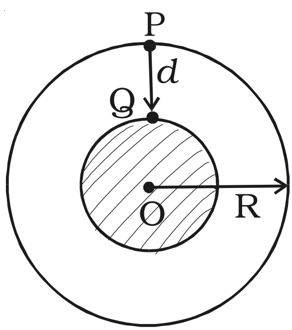


Fig. 4.5 Variation of g with depth

Let P be a point on the surface of the Earth and Q be a point at a depth d from the surface.

The acceleration due to gravity at P on the surface is $g = \frac{GM}{R^2}$.

If ρ be the density, then, the mass of the Earth is $M = \frac{4}{3}\pi R^3\rho$

$$\therefore\ g = \frac{4}{3} G \pi R \rho \qquad \dots (1)$$

The acceleration due to gravity at Q at a depth d from the surface of the Earth is

$$g_d = \frac{GM_d^{\,2}}{(R-d)^2}$$

where M_d is the mass of the inner sphere of the Earth of radius $(R- d)$.

$$M_d = \frac{4}{3}\ \pi (R - d)^3 \rho$$

$$\therefore\ g_d = \frac{4}{3} G \pi (R - d) \rho \qquad \dots (2)$$

dividing (2) by (1), $\frac{g_d}{g} = \frac{R - d}{R}$

$$g_d = g \left(1 - \frac{d}{R}\right)$$

The value of acceleration due to gravity decreases with increase of depth.

(iii) Variation of g with latitude (Non–sphericity of the Earth)

The Earth is not a perfect sphere. It is an ellipsoid as shown in the Fig. 4.6. It is flattened at the poles where the latitude is 90° and bulged at the equator where the latitude is 0°.

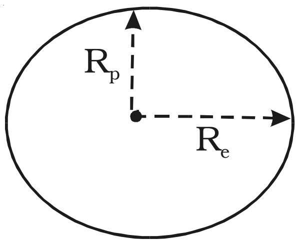


Fig.4.6 Non–sphericity of the Earth

The radius of the Earth at equatorial plane R_e is greater than the radius along the poles R_p by about 21 km.

We know that $g = \frac{GM}{R^2}$

$$\therefore\ g \ \alpha\ \frac{1}{R^2}$$

The value of g varies inversely as the square of radius of the Earth. The radius at the equator is the greatest. Hence the value of g

is minimum at the equator. The radius at poles is the least. Hence, the value of g is maximum at the poles. The value of g increases from the equator to the poles.

(iv) Variation of g with latitude (Rotation of the Earth)

Let us consider the Earth as a homogeneous sphere of mass M and radius R. The Earth rotates about an axis passing through its north and south poles. The Earth rotates from west to east in 24 hours. Its angular velocity is 7.3×10^{-5} rad s^{-1}.

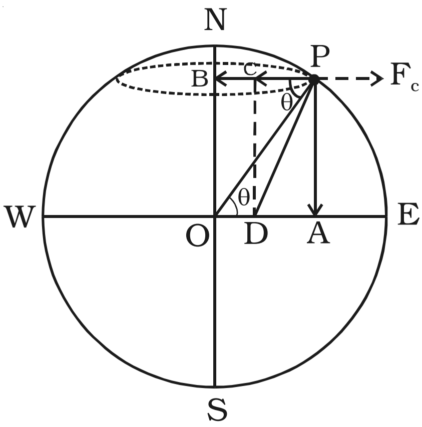


Fig. 4.7 Rotation of the Earth

Consider a body of mass m on the surface of the Earth at P at a latitude θ. Let ω be the angular velocity. The force (weight) F = mg acts along PO. It could be resolved into two rectangular components (i) mg cos θ along PB and (ii) mg sin θ along PA (Fig. 4.7).

From the ΔOPB, it is found that $BP = R \cos \theta$. The particle describes a circle with B as centre and radius $BP = R \cos \theta$.

The body at P experiences a centrifugal force (outward force) F_C due to the rotation of the Earth.

(i.e) $F_C = mR\omega^2 \cos \theta$.

The net force along $PC = mg \cos \theta - mR\omega^2 \cos \theta$

∴ The body is acted upon by two forces along PA and PC.

The resultant of these two forces is

$$F = \sqrt{(mg \sin\theta)^2 + (mg \cos\theta - mR\omega^2 \cos\theta)^2}$$

$$F = mg \sqrt{1 - \frac{2R\omega^2 \cos^2\theta}{g} + \frac{R^2\omega^4 \cos^2\theta}{g^2}}$$

since $\frac{R^2\omega^4}{g^2}$ is very small, the term $\frac{R^2\omega^4 \cos^2\theta}{g^2}$ can be neglected.

$$\text{The force, } F = mg \sqrt{1 - \frac{2R\omega^2 \cos^2\theta}{g}} \qquad \text{... (1)}$$

If g' is the acceleration of the body at P due to this force F, we have, $F = mg'$ ... (2)

by equating (2) and (1)

$$mg' = mg\sqrt{1 - \frac{2R\omega^2 \cos^2\theta}{g}}$$

$$g' = g\left(1 - \frac{R\omega^2 \cos^2\theta}{g}\right)$$

Case (i) At the poles, $\theta = 90^o$; $\cos\theta = 0$

$\therefore\ g' = g$

Case (ii) At the equator, $\theta = 0$; $\cos\theta = 1$

$$\therefore\ g' = g\left(1 - \frac{R\omega^2}{g}\right)$$

So, the value of acceleration due to gravity is maximum at the poles.

4.5 Gravitational field

Two masses separated by a distance exert gravitational forces on one another. This is called action at–a–distance. They interact even though they are not in contact. This interaction can also be explained with the field concept. *A particle or a body placed at a point modifies a space around it which is called gravitational field.* When another particle is brought in this field, it experiences gravitational force of attraction. *The gravitational field is defined as the space around a mass in which it can exert gravitational force on other mass.*

4.5.1 Gravitational field intensity

Gravitational field intensity or strength at a point is defined as the force experienced by a unit mass placed at that point. It is denoted by **E.** It is a vector quantity. Its unit is N kg^{-1}.

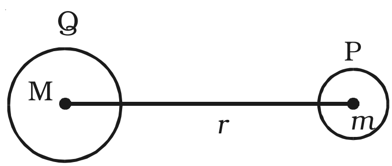


Fig. 4.8 Gravitational field

Consider a body of mass M placed at a point Q and another body of mass m placed at P at a distance r from Q.

The mass M develops a field E at P and this field exerts a force $F = mE$.

The gravitational force of attraction between the masses m and M is $F = \frac{GMm}{r^2}$

The gravitational field intensity at P is $E = \frac{F}{m}$

$$\therefore \quad E = \frac{GM}{r^2}$$

Gravitational field intensity is the measure of gravitational field.

4.5.2 Gravitational potential difference

Gravitational potential difference between two points is defined as the amount of work done in moving unit mass from one point to another point against the gravitational force of attraction.

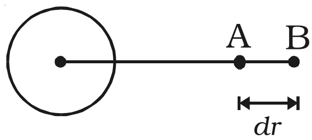


Fig. 4.9 Gravitational potential difference

Consider two points A and B separated by a distance dr in the gravitational field.

The work done in moving unit mass from A to B is $dv = W_{A \to B}$

Gravitational potential difference $dv = -E\,dr$

Here negative sign indicates that work is done against the gravitational field.

4.5.3 Gravitational potential

Gravitational potential at a point is defined as the amount of work done in moving unit mass from the point to infinity against the gravitational field. It is a scalar quantity. Its unit is $\mathrm{N\,m\,kg^{-1}}$.

4.5.4 Expression for gravitational potential at a point

Consider a body of mass M at the point C. Let P be a point at a distance r from C. To calculate the gravitational potential at P consider two points A and B. The point A, where the unit mass is placed is at a distance x from C.

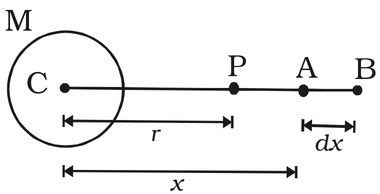


Fig. 4.10 Gravitational potential

The gravitational field at A is $E = \frac{GM}{x^2}$

The work done in moving the unit mass from A to B through a small distance dx is $dw = dv = -E.dx$

Negative sign indicates that work is done against the gravitational field.

$$dv = -\frac{GM}{x^2}\ dx$$

The work done in moving the unit mass from the point P to infinity is $\int dv = -\int_r^\infty \frac{GM}{x^2}\ dx$

$$v = -\frac{GM}{r}$$

The gravitational potential is negative, since the work is done against the field. (i.e) the gravitational force is always attractive.

4.5.5 Gravitational potential energy

Consider a body of mass m placed at P at a distance r from the centre of the Earth. Let the mass of the Earth be M.

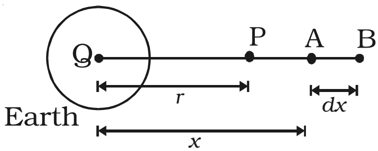


Fig. 4.11 Gravitational potential energy

When the mass m is at A at a distance x from Q, the gravitational force of attraction on it due to mass M is given by $F = \frac{GMm}{x^2}$

The work done in moving the mass m through a small distance dx from A to B along the line joining the two centres of masses m and M is $dw = -F.dx$

Negative sign indicates that work is done against the gravitational field.

$$\therefore\ dw = -\frac{GMm}{x^2} \cdot dx$$

The gravitational potential energy of a mass m at a distance r from another mass M is defined as the amount of work done in moving the mass m from a distance r to infinity.

The total work done in moving the mass m from a distance r to infinity is

$$\int dw = -\int_r^\infty \frac{GMm}{x^2} dx$$

$$W = -GMm \int_r^\infty \frac{1}{x^2} dx$$

$$*U = -\frac{GMm}{r}$$

Gravitational potential energy is zero at infinity and decreases as the distance decreases. This is due to the fact that the gravitational force exerted on the body by the Earth is attractive. Hence the gravitational potential energy U is negative.

4.5.6 Gravitational potential energy near the surface of the Earth

Let the mass of the Earth be M and its radius be R. Consider a point A on the surface of the Earth and another point B at a height h above the surface of the Earth. The work done in moving the mass m from A to B is $U = U_B - U_A$

$$U = -GMm\left[\frac{1}{(R+h)} - \frac{1}{R}\right]$$

$$U = GMm\left[\frac{1}{R} - \frac{1}{(R+h)}\right]$$

$$U = \frac{GMmh}{R(R+h)}$$

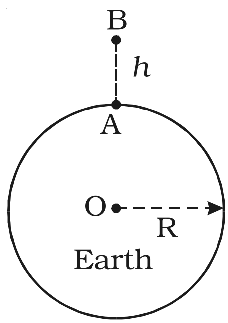


Fig. 4.12 Gravitational potential energy near the surface of the Earth

If the body is near the surface of the Earth, h is very small when compared with R. Hence $(R+h)$ could be taken as R.

$$\therefore \quad U = \frac{GMmh}{R^2}$$

$$U = mgh \qquad \left(\because \frac{GM}{R^2} = g\right)$$

4.6 Inertial mass

According to Newton's second law of motion ($F = ma$), the mass of a body can be determined by measuring the acceleration produced in it

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* *Potential energy is represented by U (Upsilon).*

by a constant force. (i.e) $m = F/a$. *Intertial mass of a body is a measure of the ability of a body to oppose the production of acceleration in it by an external force.*

If a constant force acts on two masses m_A and m_B and produces accelerations a_A and a_B respectively, then, $F = m_A a_A = m_B a_B$

$$\therefore \frac{m_A}{m_B} = \frac{a_B}{a_A}$$

The ratio of two masses is independent of the constant force. If the same force is applied on two different bodies, the inertial mass of the body is more in which the acceleration produced is less.

If one of the two masses is a standard kilogram, the unknown mass can be determined by comparing their accelerations.

4.7 Gravitational mass

According to Newton's law of gravitation, the gravitational force on a body is proportional to its mass. We can measure the mass of a body by measuring the gravitational force exerted on it by a massive body like Earth. *Gravitational mass is the mass of a body which determines the magnitude of gravitational pull between the body and the Earth.* This is determined with the help of a beam balance.

If F_A and F_B are the gravitational forces of attraction on the two bodies of masses m_A and m_B due to the Earth, then

$$F_A = \frac{G m_A M}{R^2} \text{ and } F_B = \frac{G m_B M}{R^2}$$

where M is mass of the Earth, R is the radius of the Earth and G is the gravitational constant.

$$\therefore \frac{m_A}{m_B} = \frac{F_A}{F_B}$$

If one of the two masses is a standard kilogram, the unknown mass can be determined by comparing the gravitational forces.

4.8 Escape speed

If we throw a body upwards, it reaches a certain height and then falls back. This is due to the gravitational attraction of the Earth. If we throw the body with a greater speed, it rises to a greater height. If the body is projected with a speed of 11.2 km/s, it escapes from the Earth and never comes back. *The escape speed is the minimum speed with which a body must be projected in order that it may escape from the gravitational pull of the planet.*

Consider a body of mass m placed on the Earth's surface. The gravitational potential energy is $E_P = -\frac{GMm}{R}$

where M is the mass of the Earth and R is its radius.

If the body is projected up with a speed v_e, the kinetic energy is

$$E_K = \frac{1}{2} m v_e^2$$

∴ the initial total energy of the body is

$$E_i = \frac{1}{2} m v_e^2 - \frac{GMm}{R} \quad \text{... (1)}$$

If the body reaches a height h above the Earth's surface, the gravitational potential energy is

$$E_P = -\frac{GMm}{(R+h)}$$

Let the speed of the body at the height is v, then its kinetic energy is,

$$E_K = \frac{1}{2} m v^2.$$

Hence, the final total energy of the body at the height is

$$E_f = \frac{1}{2} m v^2 - \frac{GMm}{(R+h)} \quad \text{... (2)}$$

We know that the gravitational force is a conservative force and hence the total mechanical energy must be conserved.

$$\therefore E_i = E_f$$

(i.e) $$\frac{m v_e^2}{2} - \frac{GMm}{R} = \frac{m v^2}{2} - \frac{GMm}{(R+h)}$$

The body will escape from the Earth's gravity at a height where the gravitational field ceases out. (i.e) $h = \infty$. At the height $h = \infty$, the speed v of the body is zero.

Thus $$\frac{mv_e^2}{2} - \frac{GMm}{R} = 0$$

$$v_e = \sqrt{\frac{2GM}{R}}$$

From the relation $g = \frac{GM}{R^2}$, we get $GM = gR^2$

Thus, the escape speed is $v_e = \sqrt{2gR}$

The escape speed for Earth is 11.2 km/s, for the planet Mercury it is 4 km/s and for Jupiter it is 60 km/s. The escape speed for the moon is about 2.5 km/s.

4.8.1 An interesting consequence of escape speed with the atmosphere of a planet

We know that the escape speed is independent of the mass of the body. Thus, molecules of a gas and very massive rockets will require the same initial speed to escape from the Earth or any other planet or moon.

The molecules of a gas move with certain average velocity, which depends on the nature and temperature of the gas. At moderate temperatures, the average velocity of oxygen, nitrogen and carbon–di–oxide is in the order of 0.5 km/s to 1 km/s and for lighter gases hydrogen and helium it is in the order of 2 to 3 km/s. It is clear that the lighter gases whose average velocities are in the order of the escape speed, will escape from the moon. The gravitational pull of the moon is too weak to hold these gases. The presence of lighter gases in the atmosphere of the Sun should not surprise us, since the gravitational attraction of the sun is very much stronger and the escape speed is very high about 620 km/s.

4.9 Satellites

A body moving in an orbit around a planet is called satellite. The moon is the natural satellite of the Earth. It moves around the Earth once in 27.3 days in an approximate circular orbit of radius 3.85×10^5 km. The first artificial satellite Sputnik was launched in 1956. India launched its first satellite Aryabhatta on April 19, 1975.

4.9.1 Orbital velocity

Artificial satellites are made to revolve in an orbit at a height of few hundred kilometres. At this altitude, the friction due to air is negligible. The satellite is carried by a rocket to the desired height and released horizontally with a high velocity, so that it remains moving in a nearly circular orbit.

The horizontal velocity that has to be imparted to a satellite at the determined height so that it makes a circular orbit around the planet is called orbital velocity.

Let us assume that a satellite of mass m moves around the Earth in a circular orbit of radius r with uniform speed v_o. Let the satellite be at a height h from the surface of the Earth. Hence, $r = R+h$, where R is the radius of the Earth.

The centripetal force required to keep the satellite in circular orbit is $F = \dfrac{mv_o^2}{r} = \dfrac{mv_o^2}{R+h}$

The gravitational force between the Earth and the satellite is

$$F = \frac{GMm}{r^2} = \frac{GMm}{(R+h)^2}$$

For the stable orbital motion,

$$\frac{mv_o^2}{R+h} = \frac{GMm}{(R+h)^2}$$

$$v_o = \sqrt{\frac{GM}{R+h}}$$

Since the acceleration due to gravity on Earth's surface is $g = \dfrac{GM}{R^2}$,

$$v_o = \sqrt{\frac{gR^2}{R+h}}$$

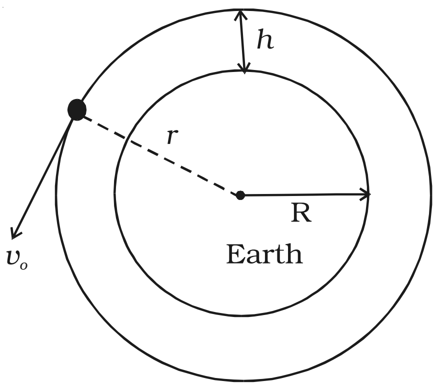


Fig. 4.13 Orbital Velocity

If the satellite is at a height of few hundred kilometres (say 200 km), $(R+h)$ could be replaced by R.

$\therefore$ orbital velocity, $v_o = \sqrt{gR}$

If the horizontal velocity (injection velocity) is not equal to the calculated value, then the orbit of the satellite will not be circular. If the injection velocity is greater than the calculated value but not greater than the escape speed ($v_e = \sqrt{2}\, v_o$), the satellite will move along an elliptical orbit. If the injection velocity exceeds the escape speed, the satellite will not revolve around the Earth and will escape into the space. If the injection velocity is less than the calculated value, the satellite will fall back to the Earth.

4.9.2 Time period of a satellite

Time taken by the satellite to complete one revolution round the Earth is called time period.

$$\text{Time period, } T = \frac{\text{circumference of the orbit}}{\text{orbital velocity}}$$

$T = \dfrac{2\pi r}{v_o} = \dfrac{2\pi(R+h)}{v_o}$ where r is the radius of the orbit which is equal to $(R+h)$.

$$T = 2\pi\,(R+h)\sqrt{\frac{R+h}{GM}} \qquad \left[\because v_o = \sqrt{\frac{GM}{R+h}}\right]$$

$$T = 2\pi\sqrt{\frac{(R+h)^3}{GM}}$$

As $GM = gR^2$, $\quad T = 2\pi\sqrt{\dfrac{(R+h)^3}{gR^2}}$

If the satellite orbits very close to the Earth, then $h << R$

$$\therefore\ T = 2\pi\sqrt{\frac{R}{g}}$$

4.9.3 Energy of an orbiting satellite

A satellite revolving in a circular orbit round the Earth possesses both potential energy and kinetic energy. If h is the height of the satellite above the Earth's surface and R is the radius of the Earth, then the radius of the orbit of satellite is $r = R+h$.

If m is the mass of the satellite, its potential energy is,

$$E_p = \frac{-GMm}{r} = \frac{-GMm}{(R+h)}$$

where M is the mass of the Earth. The satellite moves with an orbital velocity of $v_o = \sqrt{\dfrac{GM}{(R+h)}}$

Hence, its kinetic energy is, $E_K = \frac{1}{2}mv_o^2$ $E_K = \frac{GMm}{2(R+h)}$

The total energy of the satellite is, $E = E_P + E_K$

$$E = -\frac{GMm}{2(R+h)}$$

The negative value of the total energy indicates that the satellite is bound to the Earth.

4.9.4 Geo–stationary satellites

A geo-stationary satellite is a particular type used in television and telephone communications. *A number of communication satellites which appear to remain in fixed positions at a specified height above the equator are called synchronous satellites or geo-stationary satellites.* Some television programmes or events occuring in other countries are often transmitted 'live' with the help of these satellites.

For a satellite to appear fixed at a position above a certain place on the Earth, its orbital period around the Earth must be exactly equal to the rotational period of the Earth about its axis.

Consider a satellite of mass m moving in a circular orbit around the Earth at a distance r from the centre of the Earth. For synchronisation, its period of revolution around the Earth must be equal to the period of rotation of the Earth (ie) 1 day = 24 hr = 86400 seconds.

The speed of the satellite in its orbit is

$$v = \frac{\text{Circumference of orbit}}{\text{Time period}}$$

$$v = \frac{2\pi r}{T}$$

The centripetal force is $F = \frac{mv^2}{r}$

$$\therefore\ F = \frac{4m\pi^2 r}{T^2}$$

The gravitational force on the satellite due to the Earth is

$$F = \frac{GMm}{r^2}$$

For the stable orbital motion $\frac{4m\pi^2 r}{T^2} = \frac{GMm}{r^2}$ (or) $r^3 = \frac{GMT^2}{4\pi^2}$

We know that, $g = \dfrac{GM}{R^2}$

$$\therefore \ r^3 = \frac{gR^2T^2}{4\pi^2}$$

The orbital radius of the geo- stationary satellite is, $r = \left(\dfrac{gR^2T^2}{4\pi^2}\right)^{1/3}$

This orbit is called parking orbit of the satellite.

Substituting $T = 86400$ s, $R = 6400$ km and $g = 9.8$ m/s^2, the radius of the orbit of geo-stationary satellite is calculated as 42400 km.

$\therefore$ The height of the geo-stationary satellite above the surface of the Earth is $h = r - R = 36000$ km.

If a satellite is parked at this height, it appears to be stationary. Three satellites spaced at 120° intervals each above Atlantic, Pacific and Indian oceans provide a worldwide communication network.

4.9.5 Polar satellites

The polar satellites revolve around the Earth in a north–south orbit passing over the poles as the Earth spins about its north – south axis.

The polar satellites positioned nearly 500 to 800 km above the Earth travels pole to pole in 102 minutes. The polar orbit remains fixed in space as the Earth rotates inside the orbit. As a result, most of the earth's surface crosses the satellite in a polar orbit. Excellent coverage of the Earth is possible with this polar orbit. The polar satellites are used for mapping and surveying.

4.9.6 Uses of satellites

(i) Satellite communication

Communication satellites are used to send radio, television and telephone signals over long distances. These satellites are fitted with devices which can receive signals from an Earth – station and transmit them in different directions.

(ii) Weather monitoring

Weather satellites are used to photograph clouds from space and measure the amount of heat reradiated from the Earth. With this information scientists can make better forecasts about weather. You might have seen the aerial picture of our country taken by the satellites, which is shown daily in the news bulletin on the television and in the news papers.

(iii) Remote sensing

Collecting of information about an object without physical contact with the object is known as remote sensing. Data collected by the remote sensing satellities can be used in agriculture, forestry, drought assessment, estimation of crop yields, detection of potential fishing zones, mapping and surveying.

(iv) Navigation satellites

These satellites help navigators to guide their ships or planes in all kinds of weather.

4.9.7 Indian space programme

India recognised the importance of space science and technology for the socio-economic development of the society soon after the launch of Sputnik by erstwhile USSR in 1957. The Indian space efforts started in 1960 with the establishment of Thumba Equatorial Rocket Launching Station near Thiruvananthapuram for the investigation of ionosphere. The foundation of space research in India was laid by Dr. Vikram Sarabai, father of the Indian space programme. Initially, the space programme was carried out by the Department of Atomic Energy. A separate Department of Space (DOS) was established in June 1972. Indian Space Research Organisation (ISRO) under DOS executes space programme through its establishments located at different places in India (Mahendragiri in Tamil Nadu, Sriharikota in Andhra Pradesh, Thiruvananthapuram in Kerala, Bangalore in Karnataka, Ahmedabad in Gujarat, etc...). India is the sixth nation in the world to have the capability of designing, constructing and launching a satellite in an Earth orbit. The main events in the history of space research in India are given below:

Indian satellites

1. Aryabhatta - The first Indian satellite was launched on April 19, 1975.
2. Bhaskara - 1

3. Rohini

4. APPLE - It is the abbreviation of Ariane Passenger Pay Load Experiment. APPLE was the first Indian communication satellite put in geo - stationary orbit.

5. Bhaskara - 2

6. INSAT - 1A, 1B, 1C, 1D, 2A, 2B, 2C, 2D, 3A, 3B, 3C, 3D, 3E (Indian National Satellite). Indian National Satellite System is a joint venture of Department of Space, Department of Telecommunications, Indian Meteoro-logical Department and All India Radio and Doordarshan.

7. SROSS - A, B, C and D (Stretched Rohini Satellite Series)

8. IRS - 1A, 1B, 1C, 1D, P2, P3, P4, P5, P6 (Indian Remote Sensing Satellite)

Data from IRS is used for various applications like drought monitoring, flood damage assessment, flood risk zone mapping, urban planning, mineral prospecting, forest survey etc.

9. METSAT (Kalpana - I) - METSAT is the first exclusive meteorological satellite.

10. GSAT-1, GSAT-2 (Geo-stationary Satellites)

Indian Launch Vehicles (Rockets)

1. SLV - 3 - This was India's first experimental Satellite Launch Vehicle. SLV - 3 was a 22 m long, four stage vehicle weighing 17 tonne. All its stages used solid propellant.

Indian space programme is driven by the vision of Dr Vikram Sarabhai, father of the Indian Space Programme.

"There are some who question the relevance of space activities in a developing nation. To us, there is no ambiguity of purpose. We do not have the fantasy of competing with the economically advanced nations in the exploration of the moon or the planets or manned space-flight. But we are convinced that if we are to play a meaningful role nationally, and in the community of nations, we must be second to none in the application of advanced technologies to the real problems of man and society."

2. ASLV - Augmented Satellite Launch Vehicle. It was a five stage solid propellant vehicle, weighing about 40 tonnes and of about 23.8 m long.

3. PSLV - The Polar Satellite Launch Vehicle has four stages using solid and liquid propellant systems alternately. It is 44.4 m tall weighing about 294 tonnes.

4. GSLV - The Geosynchronous Satellite Launch Vehicle is a 49 m tall, three stage vehicle weighing about 414 tonnes capable of placing satellite of 1800 kg.

India's first mission to moon

ISRO has a plan to send an unmanned spacecraft to moon in the year 2008. The spacecraft is named as CHANDRAYAAN-1. This programme will be much useful in expanding scientific knowledge about the moon, upgrading India's technological capability and providing challenging opportunities for planetory research for the younger generation. This journey to moon will take 5½ days.

CHANDRAYAAN - 1 will probe the moon by orbiting it at the lunar orbit of altitude 100 km. This mission to moon will be carried by PSLV rocket.

4.9.8 Weightlessness

Television pictures and photographs show astronauts and objects floating in satellites orbiting the Earth. This apparent weightlessness is sometimes explained wrongly as zero–gravity condition. Then, what should be the reason?

Consider the astronaut standing on the ground. He exerts a force (his weight) on the ground. At the same time, the ground exerts an equal and opposite force of reaction on the astronaut. Due to this force of reaction, he has a feeling of weight.

When the astronaut is in an orbiting satellite, both the satellite and astronaut have the same acceleration towards the centre of the Earth. Hence, the astronaut does not exert any force on the floor of the satellite. So, the floor of the satellite also does not exert any force of reaction on the astronaut. As there is no reaction, the astronaut has a feeling of weightlessness.

4.9.9 Rockets – principle

A rocket is a vehicle which propels itself by ejecting a part of its mass. Rockets are used to carry the payloads (satellites). We have heard of the PSLV and GSLV rockets. All of them are based on Newton's third law of motion.

Consider a hollow cylindrical vessel closed on both ends with a small hole at one end, containing a mixture of combustible fuels (Fig. 4.14). If the fuel is ignited, it is converted into a gas under high pressure. This high pressure pushes the gas through the hole with an enormous force. This force represents the action A. Hence an opposite force, which is the reaction R, will act on the vessel and make it to move forward.

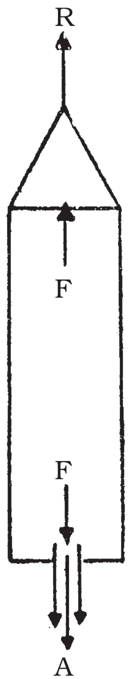


Fig. 4.14 Principle of Rocket

The force (F_m) on the escaping mass of gases and hence the rocket is proportional to the product of the mass of the gases discharged per unit time $\left(\frac{dm}{dt}\right)$ and the velocity with which they are expelled (v)

$$\text{(i.e)} \qquad F_m \ \alpha \ \frac{dm}{dt} \ v \qquad \left[\because F \alpha \frac{d}{dt}(mv)\right]$$

This force is known as momentum thrust. If the pressure (P_e) of the escaping gases differs from the pressure (P_o) in the region outside the rocket, there is an additional thrust called the velocity thrust (F_v) acts. It is given by $F_v = A\ (P_e - P_o)$ where A is the area of the nozzle through which the gases escape. Hence, the total thrust on the rocket is $F = F_m + F_v$

4.9.10 Types of fuels

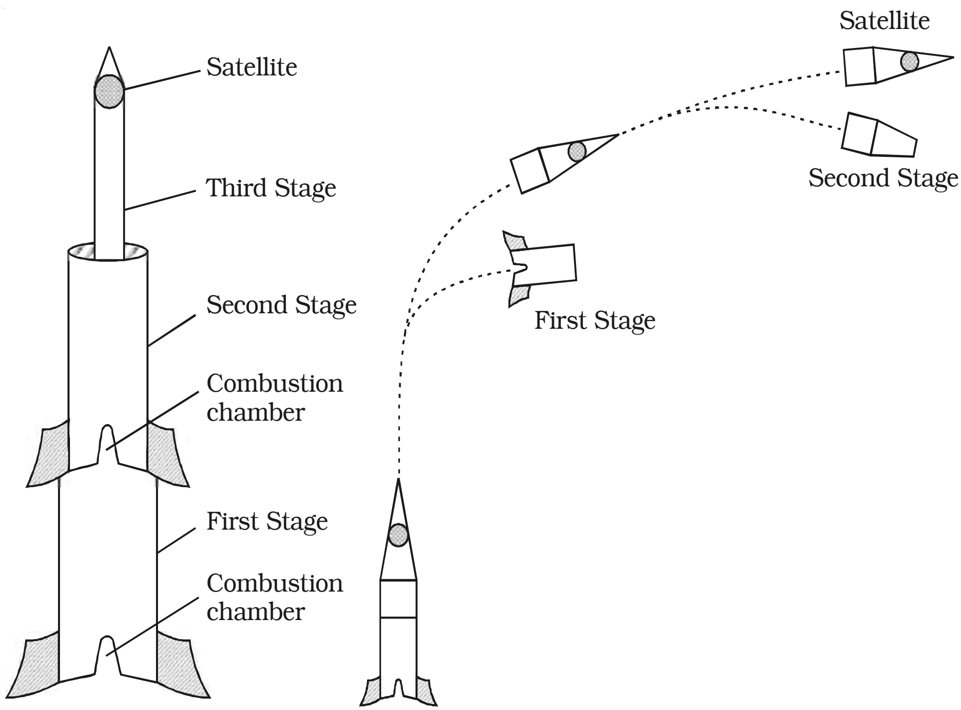
The hot gases which are produced by the combustion of a mixture of substances are called propellants. The mixture contains a fuel which burns and an oxidizer which supplies the oxygen necessary for the burning of the fuel. The propellants may be in the form of a solid or liquid.

4.9.11 Launching a satellite

To place a satellite at a height of 300 km, the launching velocity should atleast be about 8.5 km s^{-1} or 30600 kmph. If this high velocity is given to the rocket at the surface of the Earth, the rocket will be burnt due to air friction. Moreover, such high velocities cannot be developed by single rocket. Hence, multistage rockets are used.

To be placed in an orbit, a satellite must be raised to the desired height and given the correct speed and direction by the launching rocket (Fig. 4.15).

At lift off, the rocket, with a manned or unmanned satellite on top, is held down by clamps on the launching pad. Now the exhaust gases built–up an upward thrust which exceeds the rocket's weight. The clamps are then removed by remote control and the rocket accelerates upwards.



4.15 Launching a satellite

To penetrate the dense lower part of the atmosphere, initially the rocket rises vertically and then tilted by a guidance system. The first stage rocket, which may burn for about 2 minutes producing a speed of 3 km s^{-1}, lifts the vehicle to a height of about 60 km and then separates and falls back to the Earth.

The vehicle now goes to its orbital height, say 160 km, where it moves horizontally for a moment. Then the second stage of the rocket fires and increases the speed that is necessary for a circular orbit. By firing small rockets with remote control system, the satellite is separated from the second stage and made to revolve in its orbit.

4.10 The Universe

The science which deals with the study of heavenly bodies in respect of their motions, positions and compositions is known as astronomy. The Sun around which the planets revolve is a star. It is one of the hundred billion stars that comprise our galaxy called the Milky Way. A vast collection of stars held together by mutual gravitation is called a galaxy. The billions of such galaxies form the universe. Hence, the Solar system, stars and galaxies are the constituents of the universe.

4.10.1 The Solar system

The part of the universe in which the Sun occupies the central position of the system holding together all the heavenly bodies such as planets, moons, asteroids, comets ... etc., is called Solar system. The gravitational attraction of the Sun primarily governs the motion of the planets and other heavenly bodies around it. Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus, Neptune and Pluto are the nine planets that revolve around the Sun. We can see the planet Venus in the early morning in the eastern sky or in the early evening in the western sky. The planet Mercury can also be seen sometimes after the sunset in the West or just before sunrise in the East. From the Earth, the planet Mars was visibly seen on 27^{th} August 2003. The planet Mars came closer to the Earth after 60,000 years from a distance of 380×10^6 km to a nearby distance of 55.7×10^6 km. It would appear again in the year 2287.

Some of the well known facts about the solar system have been summarised in the Table 4.1.

3 m s^{-2}, 1 year = 365.257 days ; 1 AU = 1.496×10^8 km ; R_E = 6378 km ; $M_E = 5.98 \times 10^{24}$

Mass in Earth unit	Semi-major axis of orbit (AU)	Period of revolution in years	Rotation period	Mean density (kg m^{-3})	Radius in Earth unit	g in Earth unit	Escape speed (km/s)	Atmosphere	Albedo	Number of satellites
0.056	0.387	0.241	58.6 days	5,400	0.38	0.367	4	Nil	0.06	0
0.815	0.723	0.615	243 days (E → W)	5100	0.96	0.886	10.5	CO_2	0.85	0
1.000	1.000	1.000	23 hours 56.1 minutes	5520	1.00	1.000	11.2	N_2O_2	0.40	1
0.107	1.524	1.881	24 hours 27.4 minutes	3970	0.53	0.383	5	CO_2	0.15	2
0.0001	2.767	4.603	90 hours	3340	0.055	0.18	–	–	–	
317.9	5.203	11.864	9 hours 50.5 minutes	1330	11.23	2.522	60	He, CH_4, NH_3	0.45	38
95.2	9.540	29.46	10 hours 14 minutes	700	9.41	1.074	37	He, CH_4	0.61	30 +
14.6	19.18	84.01	10 hours 49 minutes (E → W)	1330	3.98	0.922	21	H_2, He, CH_4	0.35	24
17.2	30.07	164.1	15 hours	1660	3.88	1.435	22.5	H_2, He, CH_4	0.35	2
0.002	39.44	247	6.39 days	2030	0.179	0.051	1.1	–	0.14	0
0.0123	–	–	27.32 days	3340	0.27	0.170	2.5	Nil	0.07	–

4.10.2 Planetary motion

The ancient astronomers contributed a great deal by identifying the planets in the solar system and carefully plotting the variations in their positions of the sky over the periods of many years. These data eventually led to models and theories of the solar system.

The first major theory, called the Geo-centric theory was developed by a Greek astronomer, Ptolemy. The Earth is considered to be the centre of the universe, around which all the planets, the moons and the stars revolve in various orbits. The great Indian Mathematician and astronomer Aryabhat of the 5th century AD stated that the Earth rotates about its axis. Due to lack of communication between the scientists of the East and those of West, his observations did not reach the philosophers of the West.

Nicolaus Copernicus, a Polish astronomer proposed a new theory called Helio-centric theory. According to this theory, the Sun is at rest and all the planets move around the Sun in circular orbits. A Danish astronomer Tycho Brahe made very accurate observations of the motion of planets and a German astronomer Johannes Kepler analysed Brahe's observations carefully and proposed the empirical laws of planetary motion.

Kepler's laws of planetary motion

(i) The law of orbits

Each planet moves in an elliptical orbit with the Sun at one focus.

A is a planet revolving round the Sun. The position P of the planet where it is very close to the Sun is known as perigee and the position Q of the planet where it is farthest from the Sun is known as apogee.

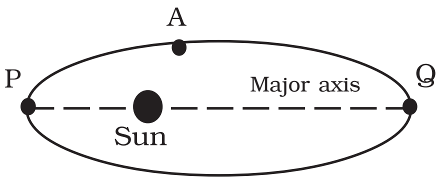


Fig. 4.16 Law of orbits

(ii) The law of areas

The line joining the Sun and the planet (i.e radius vector) sweeps out equal areas in equal interval of times.

The orbit of the planet around the Sun is as shown in Fig. 4.17. The areas A_1 and A_2 are swept by the radius vector in equal times. The planet covers unequal distances S_1 and S_2 in equal time. This is due to

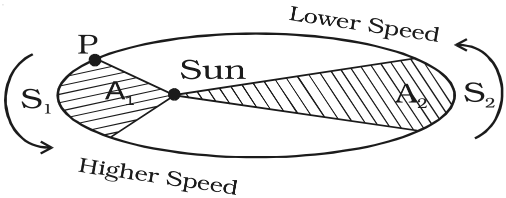


Fig. 4.17 Law of areas

the variable speed of the planet. When the planet is closest to the Sun, it covers greater distance in a given time. Hence, the speed is maximum at the closest position. When the planet is far away from the Sun, it covers lesser distance in the same time. Hence the speed is minimum at the farthest position.

Proof for the law of areas

Consider a planet moving from *A* to *B*. The radius vector *OA* sweeps a small angle $d\theta$ at the centre in a small interval of time dt.

From the Fig. 4.18, $AB = rd\,\theta$. The small area *dA* swept by the radius is,

$$dA = \frac{1}{2} \times r \times rd\theta$$

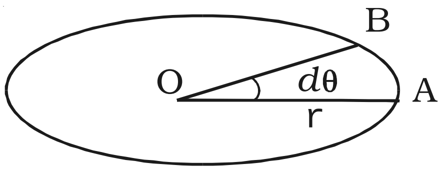


Fig. 4.18 Proof for the law of areas

Dividing by dt on both sides

$$\frac{dA}{dt} = \frac{1}{2} \times r^2 \times \frac{d\theta}{dt}$$

$$\frac{dA}{dt} = \frac{1}{2} r^2 \omega$$ where ω is the angular velocity.

The angular momentum is given by $L = mr^2\omega$

$$\therefore\ r^2\omega = \frac{L}{m}$$

Hence, $$\frac{dA}{dt} = \frac{1}{2}\frac{L}{m}$$

Since the line of action of gravitational force passes through the axis, the external torque is zero. Hence, the angular momentum is conserved.

$$\therefore\ \frac{dA}{dt} = \text{constant.}$$

(i.e) the area swept by the radius vector in unit time is the same.

(iii) The law of periods

The square of the period of revolution of a planet around the Sun is directly proportional to the cube of the mean distance between the planet and the Sun.

(i.e) $T^2 \alpha r^3$

$$\frac{T^2}{r^3} = \text{constant}$$

Proof for the law of periods

Let us consider a planet *P* of mass *m* moving with the velocity *v* around the Sun of mass *M* in a circular orbit of radius *r*.

The gravitational force of attraction of the Sun on the planet is,

$$F = \frac{GMm}{r^2}$$

The centripetal force is, $F = \frac{mv^2}{r}$

Equating the two forces

$$\frac{mv^2}{r} = \frac{GMm}{r^2}$$

$$v^2 = \frac{GM}{r} \quad .....(1)$$

If *T* be the period of revolution of the planet around the Sun, then

$$v = \frac{2\pi r}{T} \quad .....(2)$$

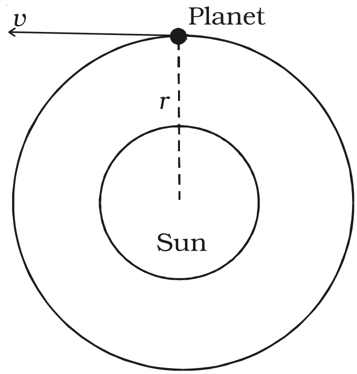


Fig. 4.19 Proof for the law of periods

Substituting (2) in (1) $\frac{4\pi^2 r^2}{T^2} = \frac{GM}{r}$

$$\frac{r^3}{T^2} = \frac{GM}{4\pi^2}$$

GM is a constant for any planet

$\therefore T^2 \alpha r^3$

4.10.3 Distance of a heavenly body in the Solar system

The distance of a planet can be accurately measured by the radar echo method. In this method, the radio signals are sent towards the planet from a radar. These signals are reflected back from the surface of a planet. The reflected signals or pulses are received and detected on

Earth. The time t taken by the signal in going to the planet and coming back to Earth is noted. The signal travels with the velocity of the light *c*. The distance *s* of the planet from the Earth is given by $s = \frac{ct}{2}$

4.10.4 Size of a planet

It is possible to determine the size of any planet once we know the distance *S* of the planet. The image of every heavenly body is a disc when viewed through a optical telescope. The angle θ between two extreme points A and B on the disc with respect to a certain point on the Earth is determined with the help of a telescope. The angle θ is called the angular diameter of the planet. The linear diameter *d* of the planet is then given by

d = distance × angular diameter

$d = s \times \theta$

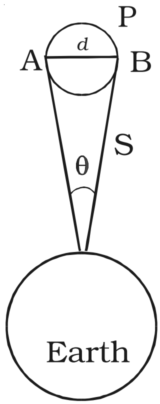


Fig. 4.20 Size of a planet

4.10.5 Surface temperatures of the planets

The planets do not emit light of their own. They reflect the Sun's light that falls on them. Only a fraction of the solar radiation is absorbed and it heats up the surface of the planet. Then it radiates energy. We can determine the surface temperature *T* of the planet using Stefan's law of radiation $E = \sigma T^4$ where σ is the Stefan's constant and *E* is the radiant energy emitted by unit area in unit time.

In general, the temperature of the planets decreases as we go away from the Sun, since the planets receive less and less solar energy according to inverse square law. Hence, the planets farther away from the Sun will be colder than those closer to it. Day temperature of Mercury is maximum (340ºC) since it is a planet closest to the Sun and that of Pluto is minimum (–240ºC). However Venus is an exception as it has very thick atmosphere of carbon–di–oxide. This acts as a blanket and keeps its surface hot. Thus the temperature of Venus is comparitively large of the order of 480ºC.

4.10.6 Mass of the planets and the Sun

In the universe one heavenly body revolves around another massive heavenly body. (The Earth revolves around the Sun and the moon revolves around the Earth). The centripetal force required by the lighter body to revolve around the heavier body is provided by the gravitational force of attraction between the two. For an orbit of given radius, the mass of the heavier body determines the speed with which the lighter body must revolve around it. Thus, if the period of revolution of the lighter body is known, the mass of the heavier body can be determined. For example, in the case of Sun – planet system, the mass of the Sun M can be calculated if the distance of the Sun from the Earth r, the period of revolution of the Earth around the Sun T and the gravitational constant G are known using the relation $M = \frac{4\pi^2}{G}\frac{r^3}{T^2}$

4.10.7 Atmosphere

The ratio of the amount of solar energy reflected by the planet to that incident on it is known as *albedo*. From the knowledge of albedo, we get information about the existence of atmosphere in the planets. The albedo of Venus is 0.85. It reflects 85% of the incident light, the highest among the nine planets. It is supposed to be covered with thick layer of atmosphere. The planets Earth, Jupiter, Saturn, Uranus and Neptune have high albedoes, which indicate that they possess atmosphere. The planet Mercury and the moon reflect only 6% of the sunlight. It indicates that they have no atmosphere, which is also confirmed by recent space probes.

There are two factors which determine whether the planets have atmosphere or not. They are (i) acceleration due to gravity on its surface and (ii) the surface temperature of the planet.

The value of g for moon is very small (¼th of the Earth). Consequently the escape speed for moon is very small. As the average velocity of the atmospheric air molecules at the surface temperature of the moon is greater than the escape speed, the air molecules escape.

Mercury has a larger value of g than moon. Yet there is no atmosphere on it. It is because, Mercury is very close to the Sun and hence its temperature is high. So the mean velocity of the gas molecules is very high. Hence the molecules overcome the gravitational attraction and escape.

4.10.8 Conditions for life on any planet

The following conditions must hold for plant life and animal life to exist on any planet.

(i) The planet must have a suitable living temperature range.

(ii) The planet must have a sufficient and right kind of atmosphere.

(iii) The planet must have considerable amount of water on its surface.

4.10.9 Other objects in the Solar system

(i) Asteroids

Asteroids are small heavenly bodies which orbit round the Sun between the orbits of Mars and Jupiter. They are the pieces of much larger planet which broke up due to the gravitational effect of Jupiter. About 1600 asteroids are revolving around the Sun. The largest among them has a diameter of about 700 km is called Ceres. It circles the Sun once in every 4½ years.

(ii) Comets

A comet consists of a small mass of rock–like material surrounded by large masses of substances such as water, ammonia and methane. These substances are easily vapourised. Comets move round the Sun in highly elliptical orbits and most of the time they keep far away from the Sun. As the comet approaches the Sun, it is heated by the Sun's radiant energy and vapourises and forms a head of about 10000 km in diameter. The comet also develops a tail pointing away from the Sun. Some comets are seen at a fixed regular intervals of time. Halley's comet is a periodic comet which made its appearance in 1910 and in 1986. It would appear again in 2062.

(iii) Meteors and Meteorites

The comets break into pieces as they approach very close to the Sun. When Earth's orbit cross the orbit of comet, these broken pieces fall on the Earth. Most of the pieces are burnt up by the heat generated due to friction in the Earth's atmosphere. They are called meteors (shooting stars). We can see these meteors in the sky on a clear moonless night.

Some bigger size meteors may survive the heat produced by friction and may not be completely burnt. These blazing objects which manage to reach the Earth are called meteorites.

The formation of craters on the surface of the moon, Mercury and Mars is due to the fact that they have been bombarded by large number of meteorites.

4.10.10 Stars

A star is a huge, more or less spherical mass of glowing gas emitting large amount of radiant energy. Billions of stars form a galaxy. There are three types of stars. They are (i) double and multiple stars (ii) intrinsically variable stars and (iii) Novae and super novae.

In a galaxy, there are only a few single stars like the Sun. Majority of the stars are either double stars (binaries) or multiple stars. The binary stars are pairs of stars moving round their common centre of gravity in stable equilibrium. An intrinsically variable star shows variation in its apparent brightness. Some stars suddenly attain extremely large brightness, that they may be seen even during daytime and then they slowly fade away. Such stars are called novae. Supernovae is a large novae.

The night stars in the sky have been given names such as Sirius (Vyadha), Canopas (Agasti), Spica (Chitra), Arcturus (Swathi), Polaris (Dhruva) ... etc. After the Sun, the star Alpha Centauri is nearest to Earth.

Sun

The Sun is extremely hot and self–luminous body. It is made of hydrogeneous matter. It is the star nearest to the Earth. Its mass is about 1.989×10^{30} kg. Its radius is about 6.95×10^{8} m. Its distance from the Earth is 1.496×10^{11} m. This is known as astronomical unit (AU). Light of the sun takes 8 minutes 20 seconds to reach the Earth. The gravitational force of attraction on the surface of the Sun is about 28 times that on the surface of the Earth.

Sun rotates about its axis from East to West. The period of revolution is 34 days at the pole and 25 days at the equator. The density of material is one fourth that of the Earth. The inner part of the Sun is a bright disc of temperature 14×10^6 K known as photosphere. The outer most layer of the Sun of temperature 6000 K is called chromosphere.

4.10.11 Constellations

Most of the stars appear to be grouped together forming interesting patterns in the sky. The configurations or groups of star formed in the patterns of animals and human beings are called constellations. There are 88 constellations into which the whole sky has been divided.

If we look towards the northern sky on a clear moonless night during the months of July and August, a group of seven bright stars resembling a bear, the four stars forming a quadrangle form the body, the remaining three stars make the tail and some other faint stars form the paws and head of the bear. This constellation is called Ursa Major or Saptarishi or Great Bear. The constellation Orion resembles the figure of a hunter and Taurus (Vrishabha) resembles the shape of a bull.

4.10.12 Galaxy

A large band of stars, gas and dust particles held together by gravitational forces is called a galaxy. Galaxies are really complex in nature consisting of billions of stars. Some galaxies emit a comparatively small amount of radio radiations compared to the total radiations emitted. They are called normal galaxies. Our galaxy Milky Way is a normal galaxy spiral in shape.

The nearest galaxy to us known as Andromeda galaxy, is also a normal galaxy. It is at a distance of 2×10^6 light years. (The distance travelled by the light in one year [9.467×10^{12} km] is called light year). Some galaxies are found to emit millions of times more radio waves compared to normal galaxies. They are called radio galaxies.

4.10.13 Milky Way galaxy

Milky Way looks like a stream of milk across the sky. Some of the important features are given below.

(i) Shape and size

Milky Way is thick at the centre and thin at the edges. The diameter of the disc is 10^5 light years. The thickness of the Milky Way varies from 5000 light years at the centre to 1000 light years at the position of the Sun and to 500 light years at the edges. The Sun is at a distance of about 27000 light years from the galactic centre.

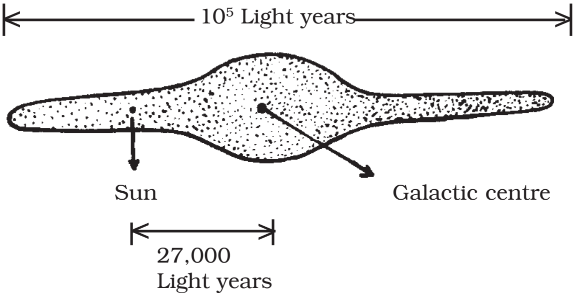


Fig. 4.21 Milky Way galaxy

(ii) Interstellar matter

The interstellar space in the Milky Way is filled with dust and gases called inter stellar matter. It is found that about 90% of the matter is in the form of hydrogen.

(iii) Clusters

Groups of stars held by mutual gravitational force in the galaxy are called star clusters. A star cluster moves as a whole in the galaxy. A group of 100 to 1000 stars is called galactic cluster. A group of about 10000 stars is called globular cluster.

(iv) Rotation

The galaxy is rotating about an axis passing through its centre. All the stars in the Milky Way revolve around the centre and complete one revolution in about 300 million years. The Sun, one of the many stars revolves around the centre with a velocity of 250 km/s and its period of revolution is about 220 million years.

(v) Mass

The mass of the Milky Way is estimated to be 3×10^{41} kg.

4.10.14 Origin of the Universe

The following three theories have been proposed to explain the origin of the Universe.

(i) Big Bang theory

According to the big bang theory all matter in the universe was concentrated as a single extremely dense and hot fire ball. An explosion occured about 20 billion years ago and the matter was broken into pieces, thrown off in all directions in the form of galaxies. Due to continuous movement more and more galaxies will go beyond the boundary and will be lost. Consequently, the number of galaxies per unit volume will go on decreasing and ultimately we will have an empty universe.

(ii) Pulsating theory

Some astronomers believe that if the total mass of the universe is more than a certain value, the expansion of the galaxies would be stopped by the gravitational pull. Then the universe may again contract. After it has contracted to a certain critical size, an explosion again occurs. The expansion and contraction repeat after every eight billion years. Thus we may have alternate expansion and contraction giving rise to a pulsating universe.

(iii) Steady state theory

According to this theory, new galaxies are continuously created out of empty space to fill up the gap caused by the galaxies which escape from the observable part of the universe. This theory, therefore suggests that the universe has always appeared as it does today and the rate of expansion has been the same in the past and will remain the same in future. So a steady state has been achieved so that the total number of galaxies in the universe remains constant.

Solved Problems

4.1 Calculate the force of attraction between two bodies, each of mass 200 kg and 2 m apart on the surface of the Earth. Will the force of attraction be different, if the same bodies are placed on the moon, keeping the separation same?

Data : $m_1 = m_2 = 200\ kg\ ;\ r = 2\ m\ ;\ G = 6.67 \times 10^{-11}\ N\,m^2\,kg^{-2}$; $F = ?$

Solution : $F = \dfrac{G\,m_1 m_2}{r^2} = \dfrac{6.67\times10^{-11}\times200\times200}{(2)^2}$

Force of attraction, $F = 6.67 \times 10^{-7}\ N$

The force of attraction on the moon will remain same, since G is the universal constant and the masses do not change.

4.2 The acceleration due to gravity at the moon's surface is 1.67 m s^{-2}. If the radius of the moon is 1.74×10^6 m, calculate the mass of the moon.

Data : $g = 1.67\ m\,s^{-2}\ ;\ R = 1.74 \times 10^6\ m\ ;$

$G = 6.67 \times 10^{-11}\ N\ m^2\ kg^{-2}\ ;\ M = ?$

Solution : $$M = \frac{gR^2}{G} = \frac{1.67\times(1.74\times10^6)^2}{6.67\times10^{-11}}$$

$$M = 7.58 \times 10^{22}\ kg$$

4.3 Calculate the height above the Earth's surface at which the value of acceleration due to gravity reduces to half its value on the Earth's surface. Assume the Earth to be a sphere of radius 6400 km.

Data : $h = ?;\ g_h = \dfrac{g}{2};\ \mathrm{R} = 6400 \times 10^3\ \mathrm{m}$

Solution : $$\frac{g_h}{g} = \frac{R^2}{(R+h)^2} = \left(\frac{R}{R+h}\right)^2$$

$$\frac{g}{2g} = \left(\frac{R}{R+h}\right)^2$$

$$\frac{R}{R+h} = \frac{1}{\sqrt{2}}$$

$$h = (\sqrt{2}-1)\, R = (1.414 - 1)\, 6400 \times 10^3$$

$$h = 2649.6 \times 10^3 \text{ m}$$

At a height of 2649.6 km from the Earth's surface, the acceleration due to gravity will be half of its value at the Earth's surface.

4.4 Determine the escape speed of a body on the moon. Given : radius of the moon is 1.74×10^6 m and mass of the moon is 7.36×10^{22} kg.

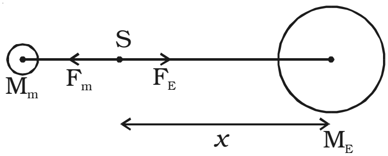
Data : $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}; $R = 1.74 \times 10^6$ m ;
$M = 7.36 \times 10^{22}$ kg; $v_e = ?$

Solution : $$v_e = \sqrt{\frac{2GM}{R}} = \sqrt{\frac{2 \times 6.67 \times 10^{-11} \times 7.36 \times 10^{22}}{1.74 \times 10^6}}$$

$$v_e = 2.375 \text{ km s}^{-1}$$

4.5 The mass of the Earth is 81 times that of the moon and the distance from the centre of the Earth to that of the moon is about 4×10^5 km. Calculate the distance from the centre of the Earth where the resultant gravitational force becomes zero when a spacecraft is launched from the Earth to the moon.

Solution :



Let the mass of the spacecraft be m. The gravitational force on the spacecraft at S due to the Earth is opposite in direction to that of the moon. Suppose the spacecraft S is at a distance x from the centre of the Earth and at a distance of $(4 \times 10^5 - x)$ from the moon.

$$\therefore \quad \frac{GM_E m}{x^2} = \frac{GM_m m}{(4 \times 10^5 - x)^2}$$

$$\frac{M_E}{M_m} = 81 = \frac{x^2}{(4 \times 10^5 - x)^2}$$

$$\therefore\ x = 3.6 \times 10^5 \text{ km.}$$

The resultant gravitational force is zero at a distance of 3.6×10^5 km from the centre of the Earth. The resultant force on S due to the Earth acts towards the Earth until 3.6×10^5 km is reached. Then it acts towards the moon.

4.6 A stone of mass 12 kg falls on the Earth's surface. If the mass of the Earth is about 6×10^{24} kg and acceleration due to gravity is 9.8 m s^{-2}, calculate the acceleration produced on the Earth by the stone.

Data : $m = 12 \text{ kg}; M = 6 \times 10^{24} \text{ kg};$

$g = a_s = 9.8 \text{ m s}^{-2}; a_E = ?$

Solution : *Let F be the gravitational force between the stone and the Earth.*

The acceleration of the stone (g) $a_S = F/m$

The acceleration of the Earth, $a_E = F/M$

$$\frac{a_E}{a_S} = \frac{m}{M} = \frac{12}{6 \times 10^{24}} = 2 \times 10^{-24}$$

$$a_E = 2 \times 10^{-24} \times 9.8$$

$$a_E = 19.6 \times 10^{-24} \text{ m s}^{-2}$$

4.7 The maximum height upto which astronaut can jump on the Earth is 0.75 m. With the same effort, to what height can he jump on the moon? The mean density of the moon is (2/3) that of the Earth and the radius of the moon is (1/4) that of the Earth.

Data : $\rho_m = \frac{2}{3} \rho_E; R_m = \frac{1}{4} R_E;$

$h_E = 0.75 \text{ m} ; h_m = ?$

Solution : *The astronaut of mass m jumps a height* h_E *on the Earth and a height* h_m *on the moon. If he gives himself the same kinetic energy on the Earth and on the moon, the potential energy gained at* h_E *and* h_m *will be the same.*

$$\therefore \quad mgh = \text{constant}$$

$$mg_m h_m = mg_E h_E$$

$$\frac{h_m}{h_E} = \frac{g_E}{g_m} \qquad \text{... (1)}$$

For the Earth, $g_E = \frac{GM_E}{R_E^2} = \frac{4}{3} \pi G R_E \rho_E$

For the moon, $g_m = \dfrac{GM_m}{R_m^{\ 2}} = \dfrac{4}{3}\ \pi G R_m \rho_m$

$$\therefore \ \frac{g_E}{g_m} = \frac{R_E}{R_m} \cdot \frac{\rho_E}{\rho_m} \qquad \text{... (2)}$$

Equating (1) and (2)

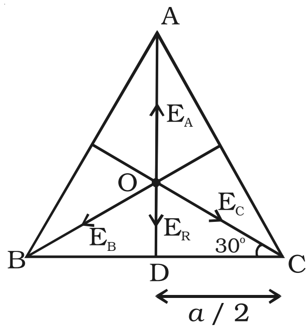
$$h_m = \frac{R_E}{R_m}\frac{\rho_E}{\rho_m} \times h_E$$

$$h_m = \frac{R_E}{\frac{1}{4}R_E} \times \frac{\rho_E}{\frac{2}{3}\rho_E} \times 0.75$$

$$h_m = 4.5 \ m$$

4.8 Three point masses, each of mass *m*, are placed at the vertices of an equilateral triangle of side *a*. What is the gravitational field and potential due to the three masses at the centroid of the triangle.

Solution :



The distance of each mass from the centroid O is OA = OB = OC

From the Δ ODC, $\cos 30^o = \dfrac{a/2}{OC}$

$$\therefore \ OC = \frac{a/2}{\cos 30^o} = a/\sqrt{3}$$

Similarly, $OB = a/\sqrt{3}$ *and* $OA = a/\sqrt{3}$

(i) *The gravitational field* $E = \dfrac{GM}{r^2}$

$\therefore$ *Field at O due to A is,* $E_A = \dfrac{3GM}{a^2}$ *(towards A)*

Field at O due to B is, $E_B = \dfrac{3GM}{a^2}$ *(towards B)*

Field at O due to C is, $E_C = \dfrac{3GM}{a^2}$ *(towards C)*

The resultant field due to E_B and E_C is

$$E_R = \sqrt{E_B^2 + E_C^2 + 2E_BE_C \cos 120^o}$$

$$E_R = \sqrt{E_B^2 + E_B^2 - E_B^2} = E_B \qquad [\because E_B = E_C]$$

The resultant field $E_R = \dfrac{3GM}{a^2}$ *acts along OD.*

Since E_A along OA and E_R along OD are equal and opposite, the net gravitational field is zero at the centroid.

(ii) *The gravitational potential is,* $v = -\dfrac{GM}{r}$

Net potential at 'O' is

$$v = -\frac{GM}{a/\sqrt{3}} - \frac{GM}{a/\sqrt{3}} - \frac{GM}{a/\sqrt{3}} = -\sqrt{3}\left(\frac{GM}{a} + \frac{GM}{a} + \frac{GM}{a}\right) = -3\sqrt{3}\,\frac{GM}{a}$$

4.9 A geo-stationary satellite is orbiting the Earth at a height of 6R above the surface of the Earth. Here R is the radius of the Earth. What is the time period of another satellite at a height of 2.5R from the surface of the Earth?

Data : *The height of the geo-stationary satellite from the Earth's surface,* $h = 6R$

The height of another satellite from the Earth's surface, $h = 2.5R$

Solution : *The time period of a satellite is* $T = 2\pi\sqrt{\dfrac{(R+h)^3}{GM}}$

$$\therefore \qquad T \ \alpha \ (R+h)^3$$

For geo-stationary satellite, $T_1 \ \alpha \ \sqrt{(R + 6R)^3}$

$$T_1 \ \alpha \ \sqrt{(7R)^3} \qquad \ldots (1)$$

For another satellite, $T_2 \ \alpha \ \sqrt{(R + 2.5R)^3}$

$$T_2 \ \alpha \ \sqrt{(3.5R)^3} \qquad \ldots (2)$$

Dividing (2) by (1) $\dfrac{T_2}{T_1} = \sqrt{\dfrac{(3.5R)^3}{(7R)^3}} = \dfrac{1}{2\sqrt{2}}$

$$T_2 = \frac{T_1}{2\sqrt{2}} = \frac{24}{2\sqrt{2}}$$

$T_2 =$ *8 hours 29 minutes* $\qquad [\because T_1 = 24$ *hours)*

5. Mechanics of Solids and Fluids

Matter is a substance, which has certain mass and occupies some volume. Matter exists in three states namely solid, liquid and gas. A fourth state of matter consisting of ionised matter of bare nuclei is called plasma. However in our forth coming discussions, we restrict ourselves to the first three states of matter. Each state of matter has some distinct properties. For example a solid has both volume and shape. It has elastic properties. A gas has the volume of the closed container in which it is kept. A liquid has a fixed volume at a given temperature, but no shape. These distinct properties are due to two factors: (i) interatomic or intermolecular forces (ii) the agitation or random motion of molecules due to temperature.

In solids, the atoms and molecules are free to vibrate about their mean positions. If this vibration increases sufficiently, molecules will shake apart and start vibrating in random directions. At this stage, the shape of the material is no longer fixed, but takes the shape of its container. This is liquid state. Due to increase in their energy, if the molecules vibrate at even greater rates, they may break away from one another and assume gaseous state. Water is the best example for this changing of states. Ice is the solid form of water. With increase in temperature, ice melts into water due to increase in molecular vibration. If water is heated, a stage is reached where continued molecular vibration results in a separation among the water molecules and therefore steam is produced. Further continued heating causes the molecules to break into atoms.

5.1 Intermolecular or interatomic forces

Consider two isolated hydrogen atoms moving towards each other as shown in Fig. 5.1.

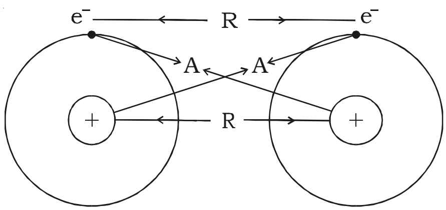


Fig. 5.1 Electrical origin of interatomic forces

As they approach each other, the following interactions are observed.

(i) Attractive force A between the nucleus of one atom and electron of the other. This attractive force tends to decrease the potential energy of the atomic system.

(ii) Repulsive force R between the nucleus of one atom and the nucleus of the other atom and electron of one atom with the electron of the other atom. These repulsive forces always tend to increase the energy of the atomic system.

There is a universal tendency of all systems to acquire a state of minimum potential energy. This stage of minimum potential energy corresponds to maximum stability.

If the net effect of the forces of attraction and repulsion leads to decrease in the energy of the system, the two atoms come closer to each other and form a covalent bond by sharing of electrons. On the other hand, if the repulsive forces are more and there is increase in the energy of the system, the atoms will repel each other and do not form a bond.

The variation of potential energy with interatomic distance between the atoms is shown in Fig. 5.2.

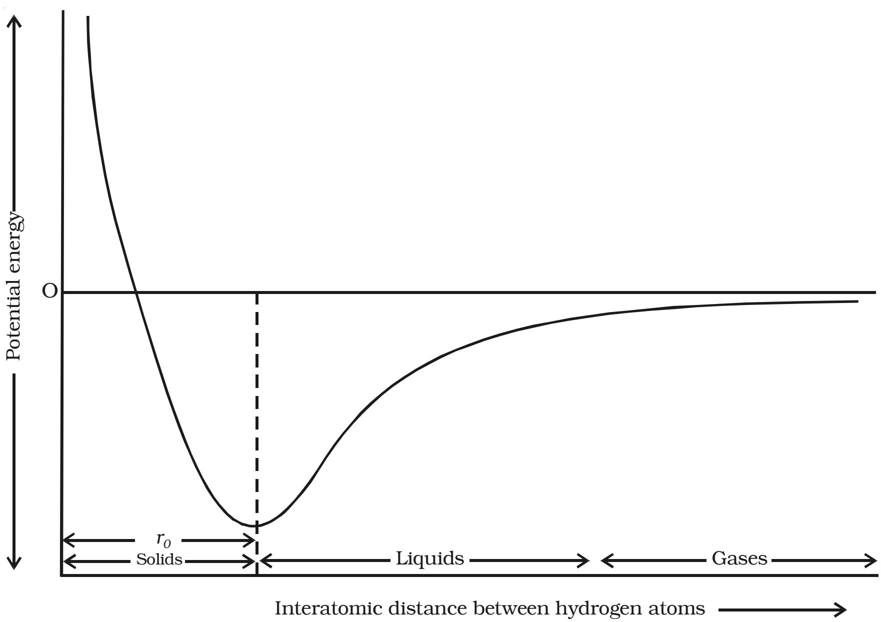


Fig. 5.2. Variation of potential energy with interatomic distance

It is evident from the graph that as the atoms come closer i.e. when the interatomic distance between them decreases, a stage is reached when the potential energy of the system decreases. When the two hydrogen atoms are sufficiently closer, sharing of electrons takes place between them and the potential energy is minimum. This results in the formation of covalent bond and the interatomic distance is r_o.

In solids the interatomic distance is r_o and in the case of liquids it is greater than r_o. For gases, it is much greater than r_o.

The forces acting between the atoms due to electrostatic interaction between the charges of the atoms are called interatomic forces. Thus, interatomic forces are electrical in nature. The interatomic forces are active if the distance between the two atoms is of the order of atomic size $\approx 10^{-10}$ *m*. In the case of molecules, the range of the force is of the order of 10^{-9} *m*.

5.2 Elasticity

When an external force is applied on a body, which is not free to move, there will be a relative displacement of the particles. Due to the property of elasticity, the particles tend to regain their original position. The external forces may produce change in length, volume and shape of the body. This external force which produces these changes in the body is called deforming force. A body which experiences such a force is called deformed body. When the deforming force is removed, the body regains its original state due to the force developed within the body. This force is called restoring force. *The property of a material to regain its original state when the deforming force is removed is called elasticity.* The bodies which possess this property are called elastic bodies. Bodies which do not exhibit the property of elasticity are called plastic. The study of mechanical properties helps us to select the material for specific purposes. For example, springs are made of steel because steel is highly elastic.

Stress and strain

In a deformed body, restoring force is set up within the body which tends to bring the body back to the normal position. The magnitude of these restoring force depends upon the deformation caused. *This restoring force per unit area of a deformed body is known as stress.*

$$\therefore \text{ Stress} = \frac{\textit{restoring force}}{\textit{area}} \; N\, m^{-2}$$

Its dimensional formula is $ML^{-1}T^{-2}$.

Due to the application of deforming force, length, volume or shape of a body changes. Or in other words, the body is said to be strained. Thus, *strain produced in a body is defined as the ratio of change in dimension of a body to the original dimension.*

$$\therefore \text{ Strain} = \frac{\textit{change in dimension}}{\textit{original dimension}}$$

Strain is the ratio of two similar quantities. Therefore it has no unit.

Elastic limit

If an elastic material is stretched or compressed beyond a certain limit, it will not regain its original state and will remain deformed. The limit beyond which permanent deformation occurs is called the elastic limit.

Hooke's law

English Physicist Robert Hooke (1635 - 1703) in the year 1676 put forward the relation between the extension produced in a wire and the restoring force developed in it. The law formulated on the basis of this study is known as Hooke's law. According to Hooke's law, *within the elastic limit, strain produced in a body is directly proportional to the stress that produces it.*

(i.e) stress α strain

$\frac{Stress}{Strain}$ = *a constant, known as modulus of elasticity.*

Its unit is N m^{-2} and its dimensional formula is $ML^{-1}T^{-2}$.

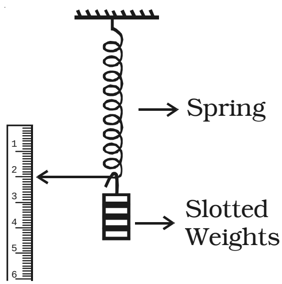


Fig. 5.3 Experimental setup to verify Hooke's law

5.2.1 Experimental verification of Hooke's law

A spring is suspended from a rigid support as shown in the Fig. 5.3. A weight hanger and a light pointer is attached at its lower end such that the pointer can slide over a scale graduated in millimeters. The initial reading on the scale is noted. A slotted weight of *m* kg is added to the weight hanger and the pointer position is noted. The same procedure is repeated with every additional *m* kg weight. It will be observed that the extension of the spring is proportional to the weight. This verifies Hooke's law.

5.2.2 Study of stress - strain relationship

Let a wire be suspended from a rigid support. At the free end, a weight hanger is provided on which weights could be added to study the behaviour of the wire under different load conditions. The extension of the wire is suitably measured and a stress - strain graph is plotted as in Fig. 5.4.

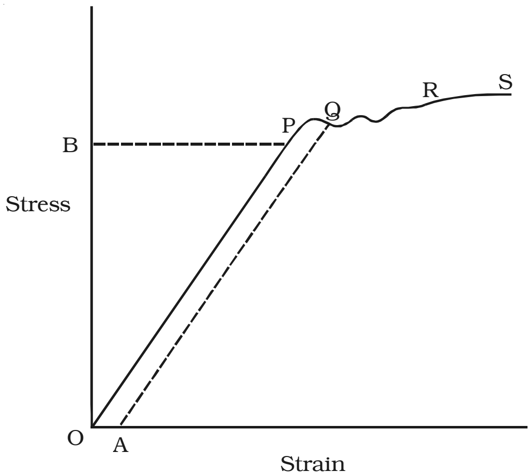


Fig. 5.4 Stress - Strain relationship

(i) In the figure the region OP is linear. Within a normal stress, strain is proportional to the applied stress. This is Hooke's law. Upto P, when the load is removed the wire regains its original length along PO. The point P represents the elastic limit, PO represents the elastic range of the material and OB is the elastic strength.

(ii) Beyond P, the graph is not linear. In the region PQ the material is partly elastic and partly plastic. From Q, if we start decreasing the load, the graph does not come to O via P, but traces a straight line QA. Thus a permanent strain OA is caused in the wire. This is called permanent set.

(iii) Beyond Q addition of even a very small load causes enormous strain. This point Q is called the yield point. The region QR is the plastic range.

(iv) Beyond R, the wire loses its shape and becomes thinner and thinner in diameter and ultimately breaks, say at S. Therefore S is the breaking point. The stress corresponding to S is called breaking stress.

5.2.3 Three moduli of elasticity

Depending upon the type of strain in the body there are three different types of modulus of elasticity. They are

(i) Young's modulus

(ii) Bulk modulus

(iii) Rigidity modulus

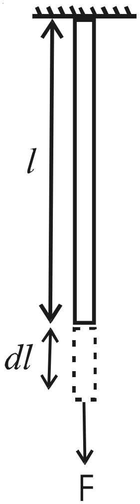


Fig. 5.5 Young's modulus of elasticity

(i) Young's modulus of elasticity

Consider a wire of length l and cross sectional area A stretched by a force F acting along its length. Let dl be the extension produced.

$$\therefore \text{ Longitudinal stress } = \frac{Force}{Area} = \frac{F}{A}$$

$$\text{Longitudinal strain } = \frac{\text{change in length}}{\text{original length}} = \frac{dl}{l}$$

Young's modulus of the material of the wire is defined as the ratio of longitudinal stress to longitudinal strain. It is denoted by q.

$$\text{Young's modulus } = \frac{\text{longitudinal stress}}{\text{longitudinal strain}}$$

$$\text{(i.e)} \quad q = \frac{F/A}{dl/l} \quad \text{or} \quad q = \frac{F\,l}{A\,dl}$$

(ii) Bulk modulus of elasticity

Suppose euqal forces act perpendicular to the six faces of a cube of volume V as shown in Fig. 5.6. Due to the action of these forces, let the decrease in volume be dV.

$$\text{Now, Bulk stress } = \frac{Force}{Area} = \frac{F}{A}$$

Bulk Strain =

$$\frac{\text{change in volume}}{\text{original volume}} = \frac{-dV}{V}$$

(The negative sign indicates that volume decreases.)

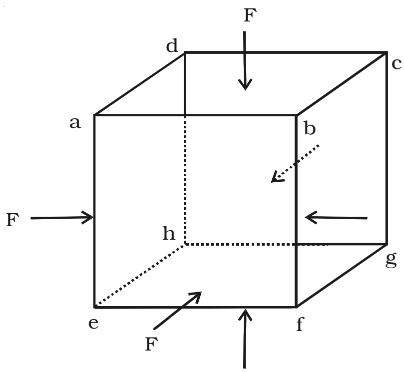


Fig. 5.6 Bulk modulus of elasticity

Bulk modulus of the material of the object is defined as the ratio bulk stress to bulk strain.

It is denoted by k.

$$\therefore \text{Bulk modulus} = \frac{\text{Bulk stress}}{\text{Bulk strain}}$$

$$\text{(i.e)} \quad k = \frac{F/A}{-\frac{dV}{V}} = \frac{P}{-\frac{dV}{V}} \quad \left[\because P = \frac{F}{A}\right] \quad \text{or} \quad k = \frac{-PV}{dV}$$

(iii) Rigidity modulus or shear modulus

Let us apply a force F tangential to the top surface of a block whose bottom AB is fixed, as shown in Fig. 5.7.

Under the action of this tangential force, the body suffers a slight change in shape, its volume remaining unchanged. The side AD of the block is sheared through an angle θ to the position AD′.

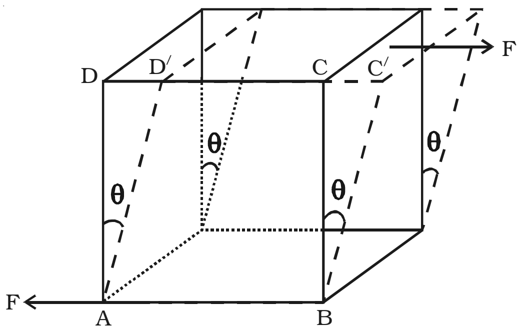


Fig. 5.7 Rigidity modulus

If the area of the top surface is A, then shear stress = F/A.

Shear modulus or rigidity modulus of the material of the object is defined as the ratio of shear stress to shear strain. It is denoted by n.

$$\textit{Rigidity modulus} = \frac{\textit{shear stress}}{\textit{shear strain}}$$

$$\text{(i.e) } n = \frac{F/A}{\theta}$$

$$= \frac{F}{A\theta}$$

Table 5.1 lists the values of the three moduli of elasticity for some commonly used materials.

Table 5.1 Values for the moduli of elasticity

Material	*Modulus of elasticity* (× 10^{11} *Pa*)		
	q	*k*	*n*
Aluminium	0.70	0.70	0.30
Copper	1.1	1.4	0.42
Iron	1.9	1.0	0.70
Steel	2.0	1.6	0.84
Tungsten	3.6	2.0	1.5

5.2.4 Relation between the three moduli of elasticity

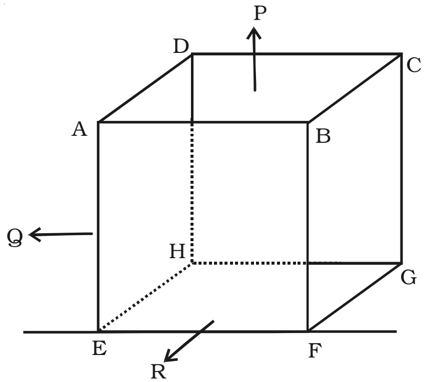


Fig. 5.8 Relation between the three moduli of elasticity

Suppose three stresses P, Q and R act perpendicular to the three faces *ABCD, ADHE* and *ABFE* of a cube of unit volume (Fig. 5.8). Each of these stresses will produce an extension in its own direction and a compression along the other two perpendicular directions. If λ is the extension per unit stress, then the elongation along the direction of P will be λP. If μ is the contraction per unit stress, then the contraction along the direction of P due to the other two stresses will be μQ and μR.

$\therefore$ The net change in dimension along the direction of P due to all the stresses is $e = \lambda P - \mu Q - \mu R$.

Similarly the net change in dimension along the direction of Q is $f = \lambda Q - \mu P - \mu R$ and the net change in dimension along the direction of R is $g = \lambda R - \mu P - \mu Q$.

Case (i)

If only P acts and $Q = R = 0$ then it is a case of longitudinal stress.

$\therefore$ Linear strain $= e = \lambda P$

$$\therefore \text{ Young's modulus } q = \frac{\textit{linear stress}}{\textit{linear strain}} = \frac{P}{\lambda P}$$

$$\text{(i.e)} \quad q = \frac{1}{\lambda} \quad \text{or} \quad \lambda = \frac{1}{q} \qquad ...(1)$$

Case (ii)

If $R = O$ and $P = -Q$, then the change in dimension along P *is* $e = \lambda P - \mu\ (-P)$

(i.e) $e = (\lambda + \mu)\ P$

Angle of shear $\theta = 2e^* = 2\ (\lambda + \mu)\ P$

$\therefore$ Rigidity modulus

$$n = \frac{P}{\theta} = \frac{P}{2(\lambda + \mu)P} \quad (or) \quad 2\ (\lambda + \mu) = \frac{1}{n} \qquad .....(2)$$

---

* *The proof for this is not given here*

Case (iii)

If $P = Q = R$, the increase in volume is $= e + f + g$

$$= 3e = 3(\lambda - 2\mu)P \quad (\text{since } e = f = g)$$

$\therefore$ Bulk strain $= 3(\lambda - 2\mu)P$

$$\text{Bulk modulus } k = \frac{P}{3(\lambda - 2\mu)P} \quad \text{or} \quad (\lambda - 2\mu) = \frac{1}{3k} \qquad ...(3)$$

$$\text{From (2), } 2(\lambda + \mu) = \frac{1}{n}$$

$$2\lambda + 2\mu = \frac{1}{n} \qquad ...(4)$$

$$\text{From (3), } (\lambda - 2\mu) = \frac{1}{3k} \qquad ...(5)$$

Adding (4) and (5),

$$3\lambda = \frac{1}{n} + \frac{1}{3k}$$

$$\lambda = \frac{1}{3n} + \frac{1}{9k}$$

$$\therefore \text{ From (1), } \frac{1}{q} = \frac{1}{3n} + \frac{1}{9k}$$

$$\text{or} \quad \frac{9}{q} = \frac{3}{n} + \frac{1}{k}$$

This is the relation between the three moduli of elasticity.

5.2.5 Determination of Young's modulus by Searle's method

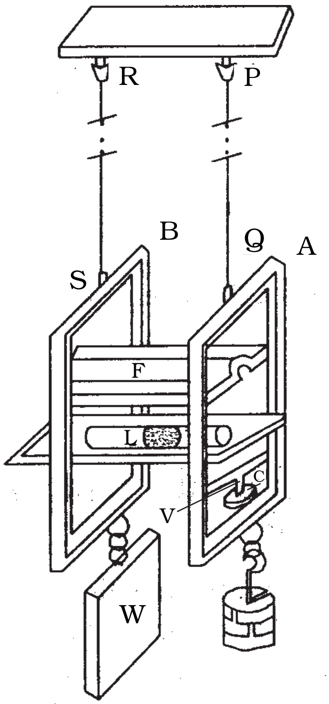


Fig. 5.9 Searle's apparatus

The Searle's apparatus consists of two rectangular steel frames A and B as shown in Fig. 5.9. The two frames are hinged together by means of a frame F. A spirit level L is provided such that one of its ends is pivoted to one of the frame B whereas the other end rests on top of a screw working through a nut in the other frame. The bottom of the screw has a circular scale C which can move along a vertical scale V graduated in mm. This vertical scale and circular scale arrangement act as pitch scale and head scale respectively of a micrometer screw.

The frames A and B are suspended from a fixed support by means of two wires PQ and RS respectively. The wire PQ attached to the frame A is the experimental wire. To keep the reference wire RS taut, a constant weight W is attached to the frame B. To the frame A, a weight hanger is attached in which slotted weights can be added.

To begin with, the experimental wire PQ is brought to the elastic mood by loading and unloading the weights in the hanger in the frame A four or five times, in steps of 0.5 kg. Then with the dead load, the micrometer screw is adjusted to ensure that both the frames are at the same level. This is done with the help of the spirit level. The reading of the micrometer is noted by taking the readings of the pitch scale and head scale. Weights are added to the weight hanger in steps of 0.5 kg upto 4 kg and in each case the micrometer reading is noted by adjusting the spirit level. The readings are again noted during unloading and are tabulated in Table 5.2. The mean extension *dl* for M kg of load is found out.

Table 5.2 Extension for M kg weight

Load in weight hanger kg	Micrometer reading			Extension for M kg weight
	Loading	Unloading	Mean	
W				
W + 0.5				
W + 1.0				
W + 1.5				
W + 2.0				
W + 2.5				
W + 3.0				
W + 3.5				
W + 4.0				

If l is the original length and r the mean radius of the experimental wire, then Young's modulus of the material of the wire is given by

$$q = \frac{F/A}{dl/l} = \frac{F/\pi r^2}{dl/l}$$

$$\text{(i.e)} \quad q = \frac{Fl}{\pi r^2 dl}$$

5.2.6 Applications of modulus of elasticity

Knowledge of the modulus of elasticity of materials helps us to choose the correct material, in right dimensions for the right application. The following examples will throw light on this.

(i) Most of us would have seen a crane used for lifting and moving heavy loads. The crane has a thick metallic rope. The maximum load that can be lifted by the rope must be specified. This maximum load under any circumstances should not exceed the elastic limit of the material of the rope. By knowing this elastic limit and the extension per unit length of the material, the area of cross section of the wire can be evaluated. From this the radius of the wire can be calculated.

(ii) While designing a bridge, one has to keep in mind the following factors (1) traffic load (2) weight of bridge (3) force of winds. The bridge is so designed that it should neither bend too much nor break.

5.3 Fluids

A fluid is a substance that can flow when external force is applied on it. The term fluids include both liquids and gases. Though liquids and gases are termed as fluids, there are marked differences between them. For example, gases are compressible whereas liquids are nearly incompressible. We only use those properties of liquids and gases, which are linked with their ability to flow, while discussing the mechanics of fluids.

5.3.1 Pressure due to a liquid column

Let h be the height of the liquid column in a cylinder of cross sectional area A. If ρ is the density of the liquid, then weight of the

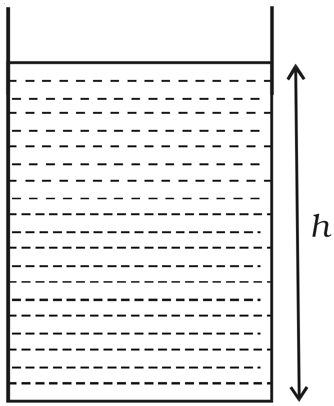


Fig. 5.10 Pressure

liquid column W is given by

W = mass of liquid column × g = $Ah\rho g$

By definition, pressure is the force acting per unit area.

$$\therefore \text{Pressure} = \frac{\text{weight of liquid column}}{\text{area of cross-section}}$$

$$= \frac{Ah\rho g}{A} = h\rho g$$

$$\therefore P = h\rho g$$

5.3.2 Pascal's law

One of the most important facts about fluid pressure is that a change in pressure at one part of the liquid will be transmitted without any change to other parts. This was put forward by Blaise Pascal (1623 - 1662), a French mathematician and physicist. This rule is known as Pascal's law.

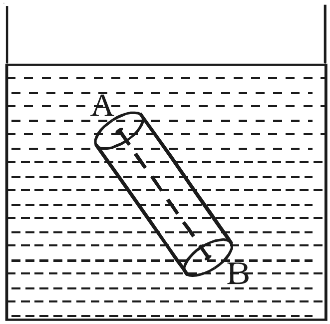


Fig. 5.11 Pascal's law in the absence of gravity

Pascal's law states that if the effect of gravity can be neglected then the pressure in a fluid in equilibrium is the same everywhere.

Consider any two points A and B inside the fluid. Imagine a cylinder such that points A and B lie at the centre of the circular surfaces at the top and bottom of the cylinder (Fig. 5.11). Let the fluid inside this cylinder be in equilibrium under the action of forces from outside the fluid. These forces act everywhere perpendicular to the surface of the cylinder. The forces acting on the circular, top and bottom surfaces are perpendicular to the forces acting on the cylindrical surface. Therefore the forces acting on the faces at A and B are equal and opposite and hence add to zero. As the areas of these two faces are equal, we can conclude that pressure at A is equal to pressure at B. This is the proof of Pascal's law when the effect of gravity is not taken into account.

Pascal's law and effect of gravity

When gravity is taken into account, Pascal's law is to be modified. Consider a cylindrical liquid column of height h and density ρ in a vessel as shown in the Fig. 5.12.

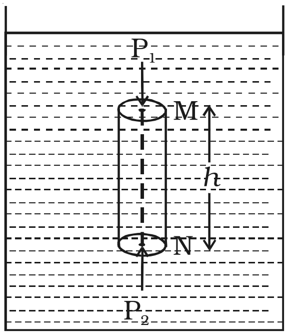


Fig. 5.12 Pascal's law and effect of gravity

If the effect of gravity is neglected, then pressure at *M* will be equal to pressure at *N*. But, if force due to gravity is taken into account, then they are not equal.

As the liquid column is in equilibrium, the forces acting on it are balanced. The vertical forces acting are

(i) Force P_1A acting vertically down on the top surface.

(ii) Weight mg of the liquid column acting vertically downwards.

(iii) Force P_2A at the bottom surface acting vertically upwards.

where P_1 and P_2 are the pressures at the top and bottom faces, A is the area of cross section of the circular face and m is the mass of the cylindrical liquid column.

At equilibrium, $P_1A + mg - P_2A = 0$ or $P_1A + mg = P_2A$

$$P_2 = P_1 + \frac{mg}{A}$$

But $m = Ah\rho$

$$\therefore \quad P_2 = P_1 + \frac{Ah\rho g}{A}$$

(i.e) $P_2 = P_1 + h\rho g$

This equation proves that the pressure is the same at all points at the same depth. This results in another statement of *Pascal's law* which can be stated as *change in pressure at any point in an enclosed fluid at rest is transmitted undiminished to all points in the fluid and act in all directions.*

5.3.3 Applications of Pascal's law

(i) Hydraulic lift

An important application of Pascal's law is the hydraulic lift used to lift heavy objects. A schematic diagram of a hydraulic lift is shown in the Fig. 5.13. It consists of a liquid container which has pistons fitted into the small and large opening cylinders. If a_1 and a_2 are the areas of the pistons A and B respectively, *F* is the force applied on A and *W* is the load on B, then

$$\frac{F}{a_1} = \frac{W}{a_2} \quad \text{or} \quad W = F\,\frac{a_2}{a_1}$$

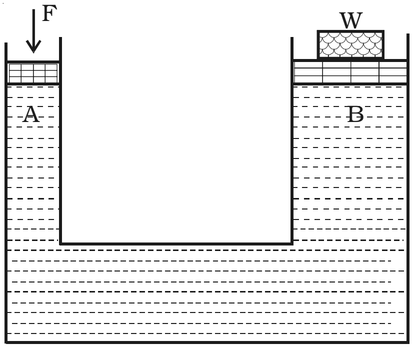


Fig. 5.13 Hydraulic lift

This is the load that can be lifted by applying a force F on A. In the above equation $\frac{a_2}{a_1}$ is called mechanical advantage of the hydraulic lift. One can see such a lift in many automobile service stations.

(ii) Hydraulic brake

When brakes are applied suddenly in a moving vehicle, there is every chance of the vehicle to skid because the wheels are not retarded uniformly. In order to avoid this danger of skidding when the brakes are applied, the brake mechanism must be such that each wheel is equally and simultaneously retarded. A hydraulic brake serves this purpose. It works on the principle of Pascal's law.

Fig. 5.14 shows the schematic diagram of a hydraulic brake system. The brake system has a main cylinder filled with brake oil. The main cylinder is provided with a piston P which is connected to the brake

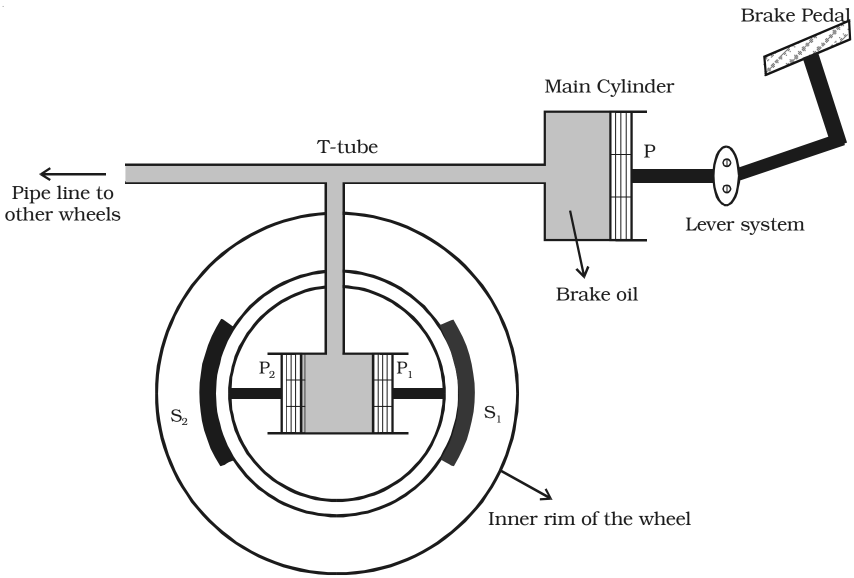


Fig. 5.14 Hydraulic brake

pedal through a lever assembly. A *T* shaped tube is provided at the other end of the main cylinder. The wheel cylinder having two pistons P_1 and P_2 is connected to the *T* tube. The pistons P_1 and P_2 are connected to the brake shoes S_1 and S_2 respectively.

When the brake pedal is pressed, piston P is pushed due to the lever assembly operation. The pressure in the main cylinder is transmitted to P_1 and P_2. The pistons P_1 and P_2 push the brake shoes away, which in turn press against the inner rim of the wheel. Thus the motion of the wheel is arrested. The area of the pistons P_1 and P_2 is greater than that of P. Therefore a small force applied to the brake pedal produces a large thrust on the wheel rim.

The main cylinder is connected to all the wheels of the automobile through pipe line for applying equal pressure to all the wheels .

5.4 Viscosity

Let us pour equal amounts of water and castor oil in two identical funnels. It is observed that water flows out of the funnel very quickly whereas the flow of castor oil is very slow. This is because of the frictional force acting within the liquid. This force offered by the adjacent liquid layers is known as viscous force and the phenomenon is called viscosity.

Viscosity is the property of the fluid by virtue of which it opposes relative motion between its different layers. Both liquids and gases exhibit viscosity but liquids are much more viscous than gases.

Co-efficient of viscosity

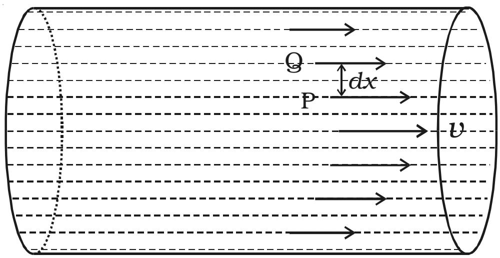


Fig. 5.15 Steady flow of a liquid

Consider a liquid to flow steadily through a pipe as shown in the Fig. 5.15. The layers of the liquid which are in contact with the walls of the pipe have zero velocity. As we move towards the axis, the velocity of the liquid layer increases and the centre layer has the maximum velocity v. Consider any two layers P and Q separated by a distance dx. Let dv be the difference in velocity between the two layers.

The viscous force F acting tangentially between the two layers of the liquid is proportional to (i) area A of the layers in contact (ii) velocity gradient $\frac{dv}{dx}$ perpendicular to the flow of liquid.

$$\therefore \quad F \,\alpha\, A \frac{dv}{dx}$$

$$F = \eta A \frac{dv}{dx}$$

where η is the coefficient of viscosity of the liquid.

This is known as Newton's law of viscous flow in fluids.

If $A = 1m^2$ and $\frac{dv}{dx} = 1s^{-1}$

then $F = \eta$

The coefficient of viscosity of a liquid is numerically equal to the viscous force acting tangentially between two layers of liquid having unit area of contact and unit velocity gradient normal to the direction of flow of liquid.

The unit of η is $N\,s\,m^{-2}$. Its dimensional formula is $ML^{-1}T^{-1}$.

5.4.1 Streamline flow

The flow of a liquid is said to be steady, streamline or laminar if every particle of the liquid follows exactly the path of its preceding particle and has the same velocity of its preceding particle at every point.

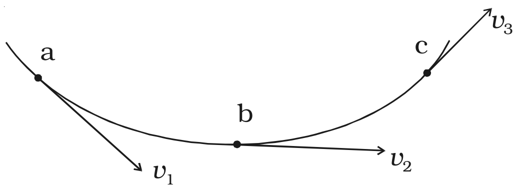


Fig. 5.16 Steamline flow

Let abc be the path of flow of a liquid and v_1, v_2 and v_3 be the velocities of the liquid at the points a, b and c respectively. During a streamline flow, all the particles arriving at 'a' will have the same velocity v_1 which is directed along the tangent at the point 'a'. A particle arriving at b will always have the same velocity v_2. This velocity v_2 may or may not be equal to v_1. Similarly all the particles arriving at the point c will always have the same velocity v_3. In other words, in the streamline flow of a liquid, the velocity of every particle crossing a particular point is the same.

The streamline flow is possible only as long as the velocity of the fluid does not exceed a certain value. This limiting value of velocity is called critical velocity.

5.4.2 *Turbulent flow*

When the velocity of a liquid exceeds the critical velocity, the path and velocities of the liquid become disorderly. At this stage, the flow loses all its orderliness and is called turbulent flow. Some examples of turbulent flow are :

(i) After rising a short distance, the smooth column of smoke from an incense stick breaks up into irregular and random patterns.

(ii) The flash - flood after a heavy rain.

Critical velocity of a liquid can be defined as that velocity of liquid upto which the flow is streamlined and above which its flow becomes turbulent.

5.4.3 *Reynold's number*

Reynolds number is a pure number which determines the type of flow of a liquid through a pipe. It is denoted by N_R.

It is given by the formula

$$N_R = \frac{v_c \, \rho \, D}{\eta}$$

where v_c is the critical velocity, ρ is the density, η is the co-efficient of viscosity of the liquid and D is the diameter of the pipe.

If N_R lies between 0 and 2000, the flow of a liquid is said to be streamline. If the value of N_R is above 3000, the flow is turbulent. If N_R lies between 2000 and 3000, the flow is neither streamline nor turbulent, it may switch over from one type to another.

Narrow tubes and highly viscous liquids tend to promote stream line motion while wider tubes and liquids of low viscosity lead to tubulence.

5.4.4 *Stoke's law (for highly viscous liquids)*

When a body falls through a highly viscous liquid, it drags the layer of the liquid immediately in contact with it. This results in a relative motion between the different layers of the liquid. As a result of this, the falling body experiences a viscous force *F*. Stoke performed many experiments on the motion of small spherical bodies in different fluids and concluded that the viscous force F acting on the spherical body depends on

(i) Coefficient of viscosity η of the liquid

(ii) Radius a of the sphere and

(iii) Velocity v of the spherical body.

Dimensionally it can be proved that

$$F = k\,\eta a v$$

Experimentally Stoke found that

$$k = 6\pi$$

$$\therefore \quad F = 6\pi\,\eta a v$$

This is Stoke's law.

5.4.5 Expression for terminal velocity

Consider a metallic sphere of radius 'a' and density ρ to fall under gravity in a liquid of density σ. The viscous force F acting on the metallic sphere increases as its velocity increases. A stage is reached when the weight W of the sphere becomes equal to the sum of the upward viscous force F and the upward thrust U due to buoyancy (Fig. 5.17). Now, there is no net force acting on the sphere and it moves down with a constant velocity v called terminal velocity.

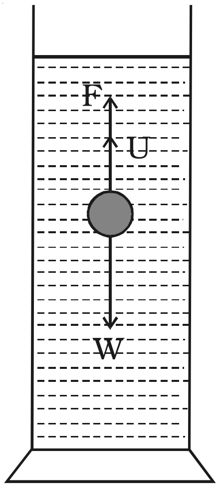


Fig. 5.17 Sphere falling in a viscous liquid

$$\therefore \; W - F - U = O \qquad \ldots(1)$$

Terminal velocity of a body is defined as the constant velocity acquired by a body while falling through a viscous liquid.

$$\text{From (1), } W = F + U \qquad \ldots(2)$$

According to Stoke's law, the viscous force F is given by $F = 6\pi\eta a v$.

The buoyant force U = Weight of liquid displaced by the sphere

$$= \frac{4}{3}\pi a^3 \sigma g$$

The weight of the sphere

$$W = \frac{4}{3}\pi a^3 \rho g$$

Substituting in equation (2),

$$\frac{4}{3}\pi a^3 \rho g = 6\pi \eta a v + \frac{4}{3}\pi a^3 \sigma g$$

$$\text{or } 6\pi \eta a v = \frac{4}{3}\pi a^3 (\rho-\sigma)g$$

$$\therefore v = \frac{2}{9}\frac{a^2(\rho-\sigma)g}{\eta}$$

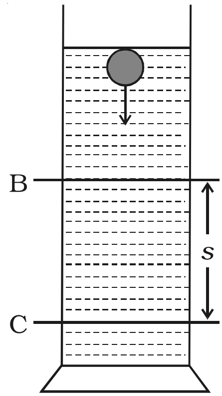


Fig. 5.18 Experimental determination of viscosity of highly viscous liquid

5.4.6 Experimental determination of viscosity of highly viscous liquids

The coefficient of highly viscous liquid like castor oil can be determined by Stoke's method. The experimental liquid is taken in a tall, wide jar. Two marking B and C are marked as shown in Fig. 5.18. A steel ball is gently dropped in the jar.

The marking B is made well below the free surface of the liquid so that by the time ball reaches B, it would have acquired terminal velocity v.

When the ball crosses B, a stopwatch is switched on and the time taken t to reach C is noted. If the distance BC is s, then terminal velocity $v = \frac{s}{t}$.

The expression for terminal velocity is

$$v = \frac{2}{9}\frac{a^2(\rho-\sigma)g}{\eta}$$

$$\therefore \frac{s}{t} = \frac{2}{9}\frac{a^2(\rho-\sigma)g}{\eta} \quad \text{or} \quad \eta = \frac{2}{9}a^2(\rho-\sigma)g\,\frac{t}{s}$$

Knowing a, ρ and σ, the value of η of the liquid is determined.

Application of Stoke's law

Falling of rain drops: When the water drops are small in size, their terminal velocities are small. Therefore they remain suspended in air in the form of clouds. But as the drops combine and grow in size, their terminal velocities increases because $v \propto a^2$. Hence they start falling as rain.

5.4.7 Poiseuille's equation

Poiseuille investigated the steady flow of a liquid through a capillary tube. He derived an expression for the volume of the liquid flowing per second through the tube.

Consider a liquid of co-efficient of viscosity η flowing, steadily through a horizontal capillary tube of length l and radius r. If P is the pressure difference across the ends of the tube, then the volume V of the liquid flowing per second through the tube depends on η, r and the pressure gradient $\left(\frac{P}{l}\right)$.

(i.e) $V \propto \eta^x r^y \left(\frac{P}{l}\right)^z$

$$V = k\, \eta^x r^y \left(\frac{P}{l}\right)^z \qquad ...(1)$$

where k is a constant of proportionality. Rewriting equation (1) in terms of dimensions,

$$[L^3T^{-1}] = [ML^{-1}\ T^{-1}]^x\ [L]^y \left[\frac{ML^{-1}T^{-2}}{L}\right]^z$$

Equating the powers of L, M and T on both sides we get $x = -1$, $y = 4$ and $z = 1$

Substituting in equation (1),

$$V = k\, \eta^{-1}\, r^4 \left(\frac{P}{l}\right)^1$$

$$V = \frac{kPr^4}{\eta l}$$

Experimentally k was found to be equal to $\frac{\pi}{8}$.

$$\therefore\ V = \frac{\pi P r^4}{8\eta l}$$

This is known as Poiseuille's equation.

5.4.8 *Determination of coefficient of viscosity of water by Poiseuille's flow method*

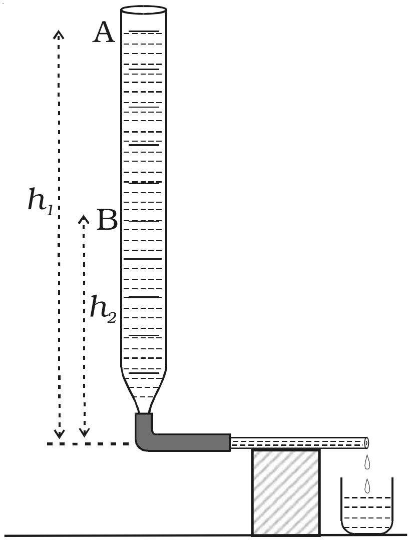


Fig. 5.19 Determination of coefficient of viscosity by Poiseuille's flow

A capillary tube of very fine bore is connected by means of a rubber tube to a burette kept vertically. The capillary tube is kept horizontal as shown in Fig. 5.19. The burette is filled with water and the pinch - stopper is removed. The time taken for water level to fall from A to B is noted. If V is the volume between the two levels A and B, then volume of liquid flowing per second is $\frac{V}{t}$. If l and r are the length and radius of the capillary tube respectively, then

$$\frac{V}{t} = \frac{\pi P r^4}{8\eta l} \qquad ...(1)$$

If ρ is the density of the liquid then the initial pressure difference between the ends of the tube is $P_1 = h_1 \rho g$ and the final pressure difference $P_2 = h_2 \rho g$. Therefore the average pressure difference during the flow of water is P where

$$P = \frac{P_1 + P_2}{2}$$

$$= \left(\frac{h_1 + h_2}{2}\right)\rho g = h\rho g \qquad \left[\because h = \frac{h_1 + h_2}{2}\right]$$

Substituting in equation (1), we get

$$\frac{V}{t} = \frac{\pi h \rho g r^4}{8 l \eta} \quad \text{or} \quad \eta = \frac{\pi h \rho g r^4 t}{8 l V}$$

5.4.9 *Viscosity - Practical applications*

The importance of viscosity can be understood from the following examples.

(i) The knowledge of coefficient of viscosity of organic liquids is used to determine their molecular weights.

(ii) The knowledge of coefficient of viscosity and its variation with temperature helps us to choose a suitable lubricant for specific machines. In light machinery thin oils (example, lubricant oil used in clocks) with low viscosity is used. In heavy machinery, highly viscous oils (example, grease) are used.

5.5 Surface tension

Intermolecular forces

The force between two molecules of a substance is called intermolecular force. This intermolecular force is basically electric in nature. When the distance between two molecules is greater, the distribution of charges is such that the mean distance between opposite charges in the molecule is slightly less than the distance between their like charges. So a force of attraction exists. When the intermolecular distance is less, there is overlapping of the electron clouds of the molecules resulting in a strong repulsive force.

The intermolecular forces are of two types. They are (i) cohesive force and (ii) adhesive force.

Cohesive force

Cohesive force is the force of attraction between the molecules of the same substance. This cohesive force is very strong in solids, weak in liquids and extremely weak in gases.

Adhesive force

Adhesive force is the force of attraction between the moelcules of two different substances. For example due to the adhesive force, ink sticks to paper while writing. Fevicol, gum etc exhibit strong adhesive property.

Water wets glass because the cohesive force between water molecules is less than the adhesive force between water and glass molecules. Whereas, mercury does not wet glass because the cohesive force between mercury molecules is greater than the adhesive force between mercury and glass molecules.

Molecular range and sphere of influence

Molecular range is the maximum distance upto which a molecule can exert force of attraction on another molecule. It is of the order of 10^{-9} m for solids and liquids.

Sphere of influence is a sphere drawn around a particular molecule as centre and molecular range as radius. The central molecule exerts a force of attraction on all the molecules lying within the sphere of influence.

5.5.1 Surface tension of a liquid

Surface tension is the property of the free surface of a liquid at rest to behave like a stretched membrane in order to acquire minimum surface area.

Imagine a line AB in the free surface of a liquid at rest (Fig. 5.20). The force of surface tension is measured as the force acting per unit length on either side of this imaginary line AB. The force is perpendicular to the line and tangential to the liquid surface. If F is the force acting on the length l of the line AB, then surface tension is given by

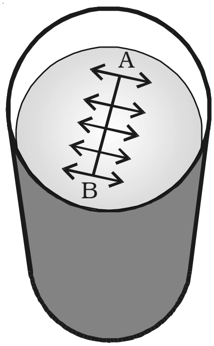


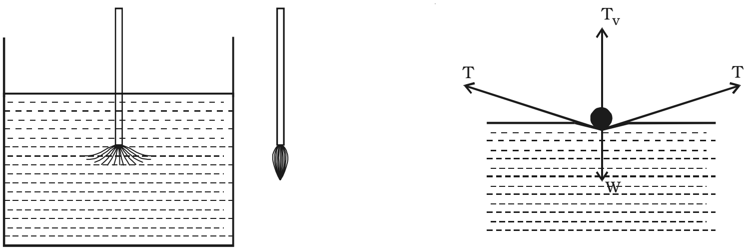
Fig. 5.20 Force on a liquid surface

$$T = \frac{F}{l}.$$

Surface tension is defined as the force per unit length acting perpendicular on an imaginary line drawn on the liquid surface, tending to pull the surface apart along the line. Its unit is $N\ m^{-1}$ and dimensional formula is MT^{-2}.

Experiments to demonstrate surface tension

(i) When a painting brush is dipped into water, its hair gets separated from each other. When the brush is taken out of water, it is observed that its hair will cling together. This is because the free surface of water films tries to contract due to surface tension.



Hair clings together when brush is taken out

Needle floats on water surface

Fig. 5.21 Practical examples for surface tension

(ii) When a sewing needle is gently placed on water surface, it floats. The water surface below the needle gets depressed slightly. The force of surface tension acts tangentially. The vertical component of the force of surface tension balances the weight of the needle.

5.5.2 Molecular theory of surface tension

Consider two molecules P and Q as shown in Fig. 5.22. Taking them as centres and molecular range as radius, a sphere of influence is drawn around them.

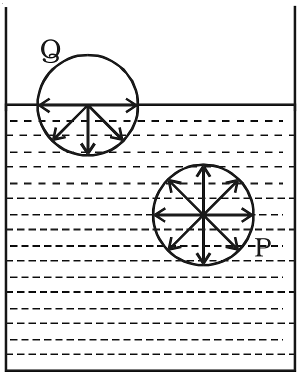


Fig. 5.22 Surface tension based on molecular theory

The molecule P is attracted in all directions equally by neighbouring molecules. Therefore net force acting on P is zero. The molecule Q is on the free surface of the liquid. It experiences a net downward force because the number of molecules in the lower half of the sphere is more and the upper half is completely outside the surface of the liquid. Therefore all the molecules lying on the surface of a liquid experience only a net downward force.

If a molecule from the interior is to be brought to the surface of the liquid, work must be done against this downward force. This work done on the molecule is stored as potential energy. *For equilibrium, a system must possess minimum potential energy.* So, the free surface will have minimum potential energy. The free surface of a liquid tends to assume minimum surface area by contracting and remains in a state of tension like a stretched elastic membrane.

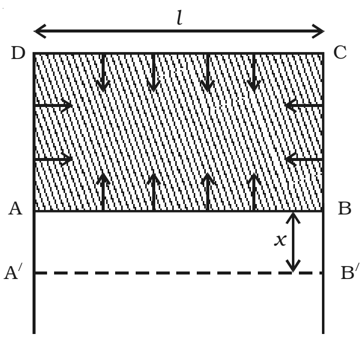


Fig. 5.23 Surface energy

5.5.3 Surface energy and surface tension

The potential energy per unit area of the surface film is called surface energy. Consider a metal frame ABCD in which AB is movable. The frame is dipped in a soap solution. A film is formed which pulls AB inwards due to surface tension. If T is the surface tension of the film and l is the length of the wire AB, this inward force is given by $2 \times Tl$. The number 2 indicates the two free surfaces of the film.

If AB is moved through a small distance x as shown in Fig. 5.23 to the position $A'B'$, then work done is

$$W = 2Tlx$$

$$\text{Work down per unit area} = \frac{W}{2lx}$$

$$\therefore \text{Surface energy} = \frac{T2lx}{2lx}$$

$$\text{Surface energy} = T$$

Surface energy is numerically equal to surface tension.

5.5.4 Angle of contact

When the free surface of a liquid comes in contact with a solid, it becomes curved at the point of contact. *The angle between the tangent to the liquid surface at the point of contact of the liquid with the solid and the solid surface inside the liquid is called angle of contact.*

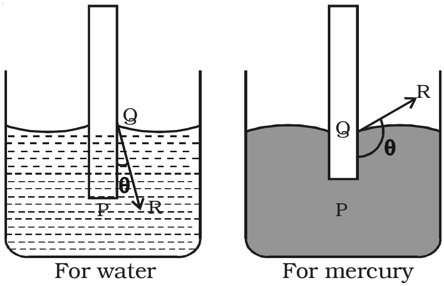


Fig. 5.24 Angle of contact

In Fig. 5.24, QR is the tangent drawn at the point of contact Q. The angle PQR is called the angle of contact. When a liquid has concave meniscus, the angle of contact is acute. When it has a convex meniscus, the angle of contact is obtuse.

The angle of contact depends on the nature of liquid and solid in contact. For water and glass, θ lies between 8^o and 18^o. For pure water and clean glass, it is very small and hence it is taken as zero. The angle of contact of mercury with glass is 138^o.

5.5.5 Pressure difference across a liquid surface

If the free surface of a liquid is plane, then the surface tension acts horizontally (Fig. 5.25a). It has no component perpendicular to the horizontal surface. As a result, there is no pressure difference between the liquid side and the vapour side.

If the surface of the liquid is concave (Fig. 5.25b), then the resultant

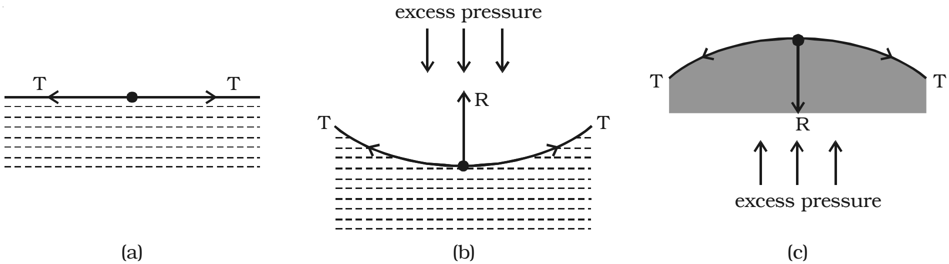


Fig. 5.25 Excess of pressure across a liquid surface

force R due to surface tension on a molecule on the surface act vertically upwards. To balance this, an excess of pressure acting downward on the concave side is necessary. On the other hand if the surface is convex (Fig. 5.25c), the resultant R acts downward and there must be an excess of pressure on the concave side acting in the upward direction.

Thus, *there is always an excess of pressure on the concave side of a curved liquid surface over the pressure on its convex side due to surface tension.*

5.5.6 Excess pressure inside a liquid drop

Consider a liquid drop of radius r. The molecules on the surface of the drop experience a resultant force acting inwards due to surface tension. Therefore, the pressure inside the drop must be greater than the pressure outside it. The excess of pressure P inside the drop provides a force acting outwards perpendicular to the surface, to balance the resultant force due to surface tension. Imagine the drop to be divided into two equal halves. Considering the equilibrium of the upper hemisphere of the drop, the upward force on the plane face ABCD due to excess pressure P is $P\pi r^2$ (Fig. 5.26).

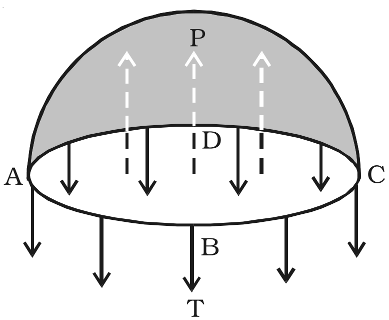


Fig. 5.26 Excess pressure inside a liquid drop

If T is the surface tension of the liquid, the force due to surface tension acting downward along the circumference of the circle ABCD is $T\,2\pi r$.

At equilibrium, $P\pi r^2 = T\,2\pi r$

$$\therefore\ P = \frac{2T}{r}$$

Excess pressure inside a soap bubble

A soap bubble has two liquid surfaces in contact with air, one inside the bubble and the other outside the bubble. Therefore the force due to surface tension = $2 \times 2\pi rT$

$$\therefore \text{ At equilibrium, } P\pi r^2 = 2 \times 2\pi rT$$

$$\text{(i.e)} \quad P = \frac{4T}{r}$$

Thus the excess of pressure inside a drop is inversely proportional to its radius i.e. $P \alpha \frac{1}{r}$. As $P \alpha \frac{1}{r}$, the pressure needed to form a very small bubble is high. This explains why one needs to blow hard to start a balloon growing. Once the balloon has grown, less air pressure is needed to make it expand more.

5.5.7 Capillarity

The property of surface tension gives rise to an interesting phenomenon called capillarity. When a capillary tube is dipped in water, the water rises up in the tube. The level of water in the tube is above the free surface of water in the beaker (capillary rise). When a capillary tube is dipped in mercury, mercury also rises in the tube. But the level of mercury is depressed below the free surface of mercury in the beaker (capillary fall).

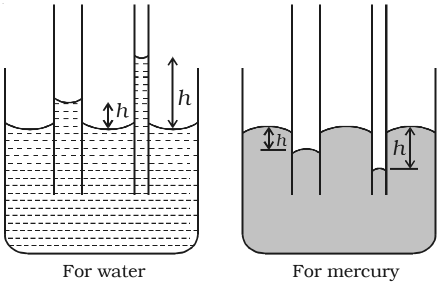


Fig. 5.27 Capillary rise

The rise of a liquid in a capillary tube is known as capillarity. The height h in Fig. 5.27 indicates the capillary rise (for water) or capillary fall (for mercury).

Illustrations of capillarity

(i) A blotting paper absorbs ink by capillary action. The pores in the blotting paper act as capillaries.

(ii) The oil in a lamp rises up the wick through the narrow spaces between the threads of the wick.

(iii) A sponge retains water due to capillary action.

(iv) Walls get damped in rainy season due to absorption of water by bricks.

5.5.8 Surface tension by capillary rise method

Let us consider a capillary tube of uniform bore dipped vertically in a beaker containing water. Due to surface tension, water rises to a height h in the capillary tube as shown in Fig. 5.28. The surface tension T of the water acts inwards and the reaction of the tube R outwards. R is equal to T in magnitude but opposite in direction. This reaction R can be resolved into two rectangular components.

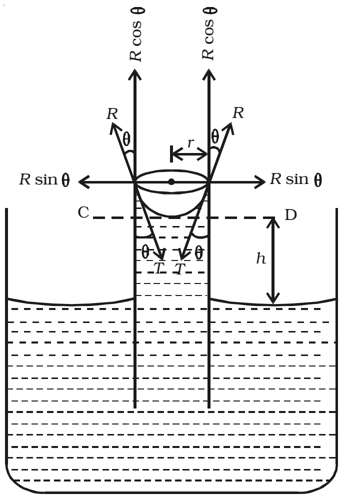


Fig. 5.28 Surface tension by capillary rise method

(i) Horizontal component $R \sin\theta$ acting radially outwards

(ii) Vertical component $R \cos\theta$ acting upwards.

The horizontal component acting all along the circumference of the tube cancel each other whereas the vertical component balances the weight of water column in the tube.

Total upward force = $R \cos\theta \times$ circumference of the tube

(i.e) $F = 2\pi r\, R \cos\theta$ or $F = 2\pi r\, T \cos\theta$ ...(1)

$[\because\ R = T]$

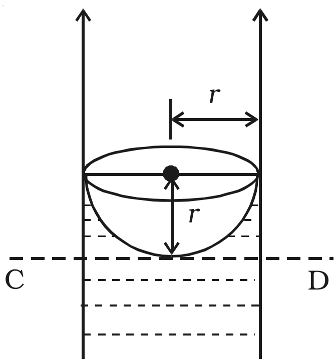


Fig. 5.29 Liquid meniscus

This upward force is responsible for the capillary rise. As the water column is in equilibrium, this force acting upwards is equal to weight of the water column acting downwards.

(i.e) $F = W$ ...(2)

Now, volume of water in the tube is assumed to be made up of (i) a cylindrical water column of height h and (ii) water in the meniscus above the plane CD.

Volume of cylindrical water column = $\pi r^2 h$

Volume of water in the meniscus = (Volume of cylinder of height r and radius r) – (Volume of hemisphere)

$\therefore$ Volume of water in the meniscus $= (\pi r^2 \times r) - \left(\frac{2}{3}\pi r^3\right)$

$$= \frac{1}{3}\pi r^3$$

$\therefore$ Total volume of water in the tube $= \pi r^2 h + \frac{1}{3}\pi r^3$

$$= \pi r^2 \left(h + \frac{r}{3}\right)$$

If ρ is the density of water, then weight of water in the tube is

$$W = \pi r^2 \left(h + \frac{r}{3}\right)\rho g \qquad ...(3)$$

Substituting (1) and (3) in (2),

$$\pi r^2 \left(h + \frac{r}{3}\right)\rho g = 2\pi r T \cos\theta$$

$$T = \frac{\left(h + \frac{r}{3}\right) r\rho g}{2\cos\theta}$$

Since r is very small, $\frac{r}{3}$ can be neglected compared to h.

$$\therefore T = \frac{hr\rho g}{2\cos\theta}$$

For water, θ is small, therefore $\cos\theta \simeq 1$

$$\therefore T = \frac{hr\rho g}{2}$$

5.5.9 *Experimental determination of surface tension of water by capillary rise method*

A clean capillary tube of uniform bore is fixed vertically with its lower end dipping into water taken in a beaker. A needle N is also fixed with the capillary tube as shown in the Fig. 5.30. The tube is raised or lowered until the tip of the needle just touches the water surface. A travelling microscope M is focussed on the meniscus of the

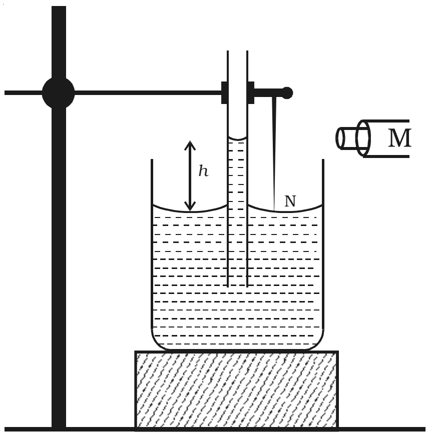


Fig. 5.30 Surface tension by capillary rise method

water in the capillary tube. The reading R_1 corresponding to the lower meniscus is noted. The microscope is lowered and focused on the tip of the needle and the corresponding reading is taken as R_2. The difference between R_1 and R_2 gives the capillary rise h.

The radius of the capillary tube is determined using the travelling microscope. If ρ is the density of water then the surface tension of water is given by $T = \frac{hr\rho g}{2}$ where g is the acceleration due to gravity.

5.5.10 Factors affecting surface tension

Impurities present in a liquid appreciably affect surface tension. A highly soluble substance like salt increases the surface tension whereas sparingly soluble substances like soap decreases the surface tension.

The surface tension decreases with rise in temperature. The temperature at which the surface tension of a liquid becomes zero is called critical temperature of the liquid.

5.5.11 Applications of surface tension

(i) During stormy weather, oil is poured into the sea around the ship. As the surface tension of oil is less than that of water, it spreads on water surface. Due to the decrease in surface tension, the velocity of the waves decreases. This reduces the wrath of the waves on the ship.

(ii) Lubricating oils spread easily to all parts because of their low surface tension.

(iii) Dirty clothes cannot be washed with water unless some detergent is added to water. When detergent is added to water, one end of the hairpin shaped molecules of the detergent get attracted to water and the other end, to molecules of the dirt. Thus the dirt is suspended surrounded by detergent molecules and this can be easily removed. This detergent action is due to the reduction of surface tension of water when soap or detergent is added to water.

(iv) Cotton dresses are preferred in summer because cotton dresses have fine pores which act as capillaries for the sweat.

5.6 Total energy of a liquid

A liquid in motion possesses pressure energy, kinetic energy and potential energy.

(i) Pressure energy

It is the energy possessed by a liquid by virtue of its pressure.

Consider a liquid of density ρ contained in a wide tank T having a side tube near the bottom of the tank as shown in Fig. 5.31. A frictionless piston of cross sectional area '*a*' is fitted to the side tube. Pressure exerted by the liquid on the piston is P = $h \rho g$ where h is the height of liquid column above the axis of the side tube. If x is the distance through which the piston is pushed inwards, then

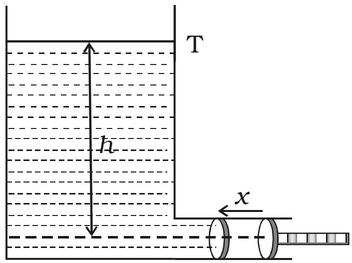


Fig. 5.31 Pressure energy

Volume of liquid pushed into the tank = ax

∴ Mass of the liquid pushed into the tank = $ax\rho$

As the tank is wide enough and a very small amount of liquid is pushed inside the tank, the height h and hence the pressure P may be considered as constant.

Work done in pushing the piston through the distance x = Force on the piston × distance moved

(i.e) $W = Pax$

This work done is the pressure energy of the liquid of mass $ax\rho$.

∴ Pressure energy per unit mass of the liquid $= \frac{Pax}{ax\rho} = \frac{P}{\rho}$

(ii) Kinetic energy

It is the energy possessed by a liquid by virtue of its motion.

If m is the mass of the liquid moving with a velocity v, the kinetic energy of the liquid $= \frac{1}{2}mv^2$.

Kinetic energy per unit mass $= \frac{\frac{1}{2}mv^2}{m} = \frac{v^2}{2}$

(iii) Potential energy

It is the energy possessed by a liquid by virtue of its height above the ground level.

If m is the mass of the liquid at a height h from the ground level, the potential energy of the liquid = mgh

Potential energy per unit mass = $\frac{mgh}{m} = gh$

Total energy of the liquid in motion = pressure energy + kinetic energy + potential energy.

$\therefore$ Total energy per unit mass of the flowing liquid = $\frac{P}{\rho} + \frac{v^2}{2} + gh$

5.6.1 Equation of continuity

Consider a non-viscous liquid in streamline flow through a tube AB of varying cross section as shown in Fig. 5.32 Let a_1 and a_2 be the area of cross section, v_1 and v_2 be the velocity of flow of the liquid at A and B respectively.

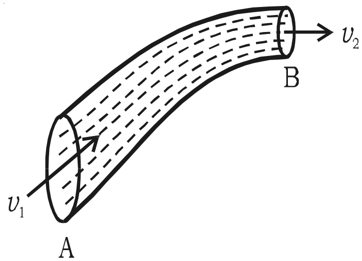


Fig. 5.32 Equation of continuity

$\therefore$ Volume of liquid entering per second at A = $a_1 v_1$.

If ρ is the density of the liquid, then mass of liquid entering per second at A = $a_1 v_1 \rho$.

Similarly, mass of liquid leaving per second at B = $a_2 v_2 \rho$

If there is no loss of liquid in the tube and the flow is steady, then mass of liquid entering per second at A = mass of liquid leaving per second at B

(i.e) $a_1 v_1 \rho = a_2 v_2 \rho$ or $a_1 v_1 = a_2 v_2$

i.e. av = constant

This is called as the equation of continuity. From this equation $v \propto \frac{1}{a}$.

i.e. the larger the area of cross section the smaller will be the velocity of flow of liquid and vice-versa.

5.6.2 Bernoulli's theorem

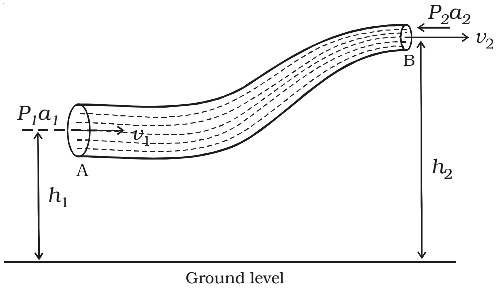


Fig. 5.33 Bernoulli's theorem

In 1738, Daniel Bernoulli proposed a theorem for the streamline flow of a liquid based on the law of conservation of energy. *According to Bernoulli's theorem, for the streamline flow of a non-viscous and incompressible liquid, the sum of the pressure energy, kinetic energy and potential energy per unit mass is a constant.*

$$\text{(i.e)} \quad \frac{P}{\rho} + \frac{v^2}{2} + gh = \text{constant}$$

This equation is known as *Bernoulli's equation.*

Consider streamline flow of a liquid of density ρ through a pipe AB of varying cross section. Let P_1 and P_2 be the pressures and a_1 and a_2, the cross sectional areas at A and B respectively. The liquid enters A normally with a velocity v_1 and leaves B normally with a velocity v_2. The liquid is accelerated against the force of gravity while flowing from A to B, because the height of B is greater than that of A from the ground level. Therefore P_1 is greater than P_2. This is maintained by an external force.

The mass m of the liquid crossing per second through any section of the tube in accordance with the equation of continuity is

$$a_1v_1\rho = a_2v_2\rho = m$$

$$\text{or} \quad a_1v_1 = a_2v_2 = \frac{m}{\rho} = V \qquad \text{..... (1)}$$

As $a_1 > a_2$, $v_1 < v_2$

The force acting on the liquid at A = P_1a_1

The force acting on the liquid at B = $P_2\ a_2$

Work done per second on the liquid at A = $P_1a_1 \times v_1 = P_1V$

Work done by the liquid at B = $P_2a_2 \times v_2 = P_2V$

∴ Net work done per second on the liquid by the pressure energy in moving the liquid from A to B is = $P_1V - P_2V$ ...(2)

If the mass of the liquid flowing in one second from A to B is m, then increase in potential energy per second of liquid from A to B is $mgh_2 - mgh_1$

Increase in kinetic energy per second of the liquid

$$= \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

According to work-energy principle, work done per second by the pressure energy = Increase in potential energy per second + Increase in kinetic energy per second

$$\text{(i.e)} \quad P_1V - P_2V = (mgh_2 - mgh_1) + \left(\frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2\right)$$

$$P_1V + mgh_1 + \frac{1}{2}mv_1^2 = P_2V + mgh_2 + \frac{1}{2}mv_2^2$$

$$\frac{P_1V}{m} + gh_1 + \frac{1}{2}v_1^2 = \frac{P_2V}{m} + gh_2 + \frac{1}{2}v_2^2$$

$$\frac{P_1}{\rho} + gh_1 + \frac{1}{2}v_1^2 = \frac{P_2}{\rho} + gh_2 + \frac{1}{2}v_2^2 \qquad \left(\because \rho = \frac{m}{v}\right)$$

or $$\frac{P}{\rho} + gh + \frac{1}{2}v^2 = \text{constant} \qquad ...(3)$$

This is Bernoulli's equation. Thus the total energy of unit mass of liquid remains constant.

Dividing equation (3) by g, $\frac{P}{\rho g} + \frac{v^2}{2g} + h = \text{constant}$

Each term in this equation has the dimension of length and hence is called head. $\frac{P}{\rho g}$ is called pressure head, $\frac{v^2}{2g}$ is velocity head and h is the gravitational head.

Special case :

If the liquid flows through a horizontal tube, $h_1 = h_2$. Therefore there is no increase in potential energy of the liquid i.e. the gravitational head becomes zero.

$\therefore$ equation (3) becomes

$$\frac{P}{\rho} + \frac{1}{2}v^2 = \text{a constant}$$

This is another form of Bernoulli's equation.

5.6.3 Application of Bernoulli's theorem

(i) Lift of an aircraft wing

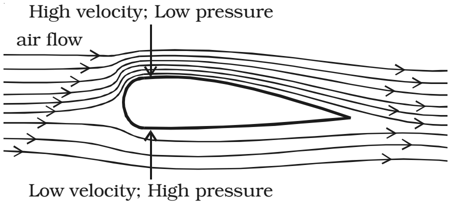


Fig. 5.34 Lift of an aircraft wing

A section of an aircraft wing and the flow lines are shown in Fig. 5.34. The orientation of the wing relative to the flow direction causes the flow lines to crowd together above the wing. This corresponds to increased velocity in this region and hence the pressure is reduced. But below the wing, the pressure is nearly equal to the atmospheric pressure. As a result of this, the upward force on the underside of the wing is greater than the downward force on the topside. Thus there is a net upward force or lift.

(ii) Blowing of roofs

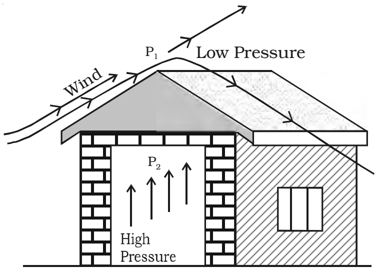


Fig. 5.35 Blowing of roofs

During a storm, the roofs of huts or tinned roofs are blown off without any damage to other parts of the hut. The blowing wind creates a low pressure P_1 on top of the roof. The pressure P_2 under the roof is however greater than P_1. Due to this pressure difference, the roof is lifted and blown off with the wind.

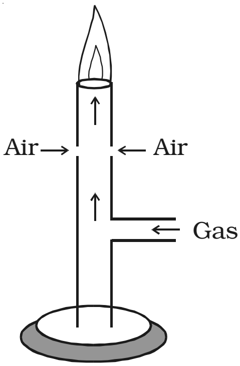


Fig. 5.36 Bunsen Burner

(iii) Bunsen burner

In a Bunsen burner, the gas comes out of the nozzle with high velocity. Due to this the pressure in the stem of the burner decreases. So, air from the atmosphere rushes into the burner.

(iv) Motion of two parallel boats

When two boats separated by a small distance row parallel to each other along the same direction, the velocity of water between the boats becomes very large compared to that on the outer sides. Because of this, the pressure in between the two boats gets reduced. The high pressure on the outer side pushes the boats inwards. As a result of this, the boats come closer and may even collide.

Solved problems

5.1 A 50 kg mass is suspended from one end of a wire of length 4 m and diameter 3 mm whose other end is fixed. What will be the elongation of the wire? Take q = 7×10^{10} N m^{-2} for the material of the wire.

Data : $l = 4$ m; $d = 3$ mm $= 3 \times 10^{-3}$ m; $m = 50$ kg; $q = 7\times10^{10}$ N m^{-2}

Solution : $q = \frac{Fl}{Adl}$

$$\therefore dl = \frac{Fl}{\pi r^2 q} = \frac{50 \times 9.8 \times 4}{3.14 \times (1.5 \times 10^{-3})^2 \times 7 \times 10^{10}}$$

$$= 3.96 \times 10^{-3} \text{ m}$$

5.2 A sphere contracts in volume by 0.01% when taken to the bottom of sea 1 km deep. If the density of sea water is 10^3 kg m^{-3}, find the bulk modulus of the material of the sphere.

Data : $dV = 0.01\%$

$$\text{i.e } \frac{dV}{V} = \frac{0.01}{100} \;;\; h = 1 \text{ km} \;;\; \rho = 10^3 \text{ kg m}^{-3}$$

Solution : $dP = 10^3 \times 10^3 \times 9.8 = 9.8 \times 10^6$

$$\therefore k = \frac{dP}{dV/V} = \frac{9.8 \times 10^6 \times 100}{0.01} = 9.8 \times 10^{10} \text{ N m}^{-2}$$

5.3 A hydraulic automobile lift is designed to lift cars with a maximum mass of 3000 kg. The area of cross–section of the piston carrying the load is 425×10^{-4} m^2. What maximum pressure would the piston have to bear?

Data : $m = 3000$ kg, $A = 425 \times 10^{-4}$ m^2

Solution: *Pressure on the piston* $= \frac{\text{Weight of car}}{\text{Area of piston}} = \frac{mg}{A}$

$$= \frac{3000 \times 9.8}{425 \times 10^{-4}} = 6.92 \times 10^5 \text{ N m}^{-2}$$

5.4 A square plate of 0.1 m side moves parallel to another plate with a velocity of 0.1 m s^{-1}, both plates being immersed in water. If the viscous force is 2×10^{-3} N and viscosity of water is 10^{-3} N s m^{-2}, find their distance of separation.

Data : *Area of plate A = 0.1 × 0.1 = 0.01 m^2*

Viscous force F = 2 × 10^{-3} N

Velocity dv = 0.1 m s^{-1}

Coefficient of viscosity η = 10^{-3} N s m^{-2}

Solution : *Distance dx* $= \dfrac{\eta A dv}{F}$

$$= \frac{10^{-3} \times 0.01 \times 0.1}{2 \times 10^{-3}} = 5 \times 10^{-4}\ m$$

5.5 Determine the velocity for air flowing through a tube of 10^{-2} m radius. For air ρ = 1.3 kg m^{-3} and η = 187 x 10^{-7} N s m^{-2}.

Data : *$r = 10^{-2}$ m ; $\rho = 1.3$ kg m^{-3} ; $\eta = 187 \times 10^{-7}$ N s m^{-2} ; $N_R = 2000$*

Solution : *velocity* $v = \dfrac{N_R \eta}{\rho D}$

$$= \frac{2000 \times 187 \times 10^{-7}}{1.3 \times 2 \times 10^{-2}} = 1.44\ m\ s^{-1}$$

5.6 Fine particles of sand are shaken up in water contained in a tall cylinder. If the depth of water in the cylinder is 0.3 m, calculate the size of the largest particle of sand that can remain suspended after 40 minutes. Assume density of sand = 2600 kg m^{-3} and viscosity of water = 10^{-3} N s m^{-2}.

Data : *s = 0.3 m, t = 40 minutes = 40 × 60 s, ρ = 2600 kg m^{-3}*

Solution: *Let us assume that the sand particles are spherical in shape and are of different size.*

Let r be the radius of the largest particle.

Terminal velocity $v = \dfrac{0.3}{40 \times 60} = 1.25 \times 10^{-4}\ m\ s^{-1}$

Radius r $= \sqrt{\dfrac{9\eta v}{2(\rho - \sigma)g}}$

$$= \sqrt{\frac{9 \times 10^{-3} \times 1.25 \times 10^{-4}}{2\,(2600 - 1000)\,9.8}}$$

$$= 5.989 \times 10^{-6}\ m$$

5.7 A circular wire loop of 0.03 m radius is rested on the surface of a liquid and then raised. The pull required is 0.003 kg wt greater than the force acting after the film breaks. Find the surface tension of the liquid.

Solution: *The additional pull F of 0.003 kg wt is the force due to surface tension.*

$\therefore$ *Force due to surface tension,*

$F = T \times$ *length of ring in contact with liquid*

(i.e) $F = T \times 2 \times 2\pi r = 4\pi Tr$

(i.e) $4\pi Tr = F$

$\therefore \; 4\pi Tr = 0.003 \times 9.81$

$$\text{or } \; T = \frac{0.003 \times 9.81}{4 \times 3.14 \times 0.03} = 0.078 \text{ N m}^{-1}$$

5.8 Calculate the diameter of a capillary tube in which mercury is depressed by 2.219 mm. Given T for mercury is 0.54 N m^{-1}, angle of contact is 140^o and density of mercury is 13600 kg m^{-3}

Data : $h = -2.219 \times 10^{-3}$ m; $T = 0.54$ N m^{-1} ; $\theta = 140^o$;

$\rho = 13600$ kg m^{-3}

Solution : $hr\rho g = 2T\cos\theta$

$$\therefore r = \frac{2T\cos\theta}{h\rho g}$$

$$= \frac{2 \times 0.54 \times \cos 140^o}{(-2.219 \times 10^{-3}) \times 13600 \times 9.8}$$

$= 2.79 \times 10^{-3}$ m

Diameter $= 2r = 2 \times 2.79 \times 10^{-3}$ m $= 5.58$ mm

5.9 Calculate the energy required to split a water drop of radius 1×10^{-3} m into one thousand million droplets of same size. Surface tension of water = 0.072 N m^{-1}

Data : *Radius of big drop* $R = 1 \times 10^{-3}$ m

Number of drops $n = 10^3 \times 10^6 = 10^9$; $T = 0.072$ N m^{-1}

Solution : *Let r be the radius of droplet.*

Volume of 10^9 drops = Volume of big drop

$$10^9 \times \frac{4}{3}\pi r^3 = \frac{4}{3}\pi R^3$$

$$10^9 r^3 = R^3 = (10^{-3})^3$$

$$(10^3 r)^3 = (10^{-3})^3$$

$$r = \frac{10^{-3}}{10^3} = 10^{-6}\ m$$

Increase in surface area $ds = 10^9 \times 4\pi r^2 - 4\pi R^2$

(i.e) $ds = 4\pi [\ 10^9 \times (10^{-6})^2 - (10^{-3})^2\ ] = 4\ \pi[10^{-3} - 10^{-6}]\ m^2$

$\therefore \quad ds = 0.01254\ m^2$

Work done $W = T.ds = 0.072 \times 0.01254 = 9.034 \times 10^{-4}\ J$

5.10 Calculate the minimum pressure required to force the blood from the heart to the top of the head (a vertical distance of 0.5 m). Given density of blood = 1040 kg m^{-3}. Neglect friction.

Data : $h_2 - h_1 = 0.5\ m$, $\rho = 1040\ kg\ m^{-3}$, $P_1 - P_2 = ?$

Solution : *According to Bernoulli's theorem*

$$P_1 - P_2 = \rho g(h_2 - h_1) + \frac{1}{2}\rho\ (v_2^2 - v_1^2)$$

If $v_2 = v_1$, *then*

$$P_1 - P_2 = \rho g\ (h_2 - h_1)$$

$$P_1 - P_2 = 1040 \times 9.8\ (0.5)$$

$$P_1 - P_2 = 5.096 \times 10^3\ N\ m^{-2}$$

6. Oscillations

Any motion that repeats itself after regular intervals of time is known as a *periodic motion*. The examples of periodic motion are the motion of planets around the Sun, motion of hands of a clock, motion of the balance wheel of a watch, motion of Halley's comet around the Sun observable on the Earth once in 76 years.

If a body moves back and forth repeatedly about a mean position, it is said to possess *oscillatory motion*. Vibrations of guitar strings, motion of a pendulum bob, vibrations of a tuning fork, oscillations of mass suspended from a spring, vibrations of diaphragm in telephones and speaker system and freely suspended springs are few examples of oscillatory motion. In all the above cases of vibrations of bodies, the path of vibration is always directed towards the mean or equilibrium position.

The oscillations can be expressed in terms of simple harmonic functions like sine or cosine function. A harmonic oscillation of constant amplitude and single frequency is called *simple harmonic motion (SHM).*

6.1 Simple harmonic motion

A particle is said to execute simple harmonic motion if its acceleration is directly proportional to the displacement from a fixed point and is always directed towards that point.

Consider a particle P executing SHM along a straight line between A and B about the mean position O (Fig. 6.1). The acceleration of the particle is always directed towards a fixed point on the line and its magnitude is proportional to the displacement of the particle from this point.

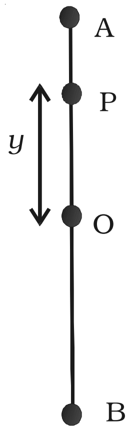


Fig. 6.1 Simple harmonic motion of a particle

(i.e) $a \propto y$

By definition $a = -\omega^2 y$

where ω is a constant known as angular frequency of the simple harmonic motion. The negative sign indicates that the acceleration is opposite to the direction of displacement. If m is the mass of the particle, restoring force that tends to bring back the particle to the mean position is given by

$$F = -m\,\omega^2\, y$$

or $$F = -k\, y$$

The constant $k = m\,\omega^2$, is called force constant or spring constant. Its unit is $N\ m^{-1}$. The restoring force is directed towards the mean position.

Thus, *simple harmonic motion is defined as oscillatory motion about a fixed point in which the restoring force is always proportional to the displacement and directed always towards that fixed point.*

6.1.1 The projection of uniform circular motion on a diameter is SHM

Consider a particle moving along the circumference of a circle of radius a and centre O, with uniform speed v, in anticlockwise direction as shown in Fig. 6.2. Let XX' and YY' be the two perpendicular diameters.

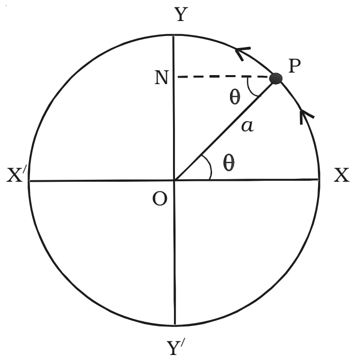


Fig. 6.2 Projection of uniform circular motion

Suppose the particle is at P after a time t. If ω is the angular velocity, then the angular displacement θ in time t is given by θ = ωt. From P draw PN perpendicular to YY'. As the particle moves from X to Y, foot of the perpendicular N moves from O to Y. As it moves further from Y to X', then from X' to Y' and back again to X, the point N moves from Y to O, from O to Y' and back again to O. When the particle completes one revolution along the circumference, the point N completes one vibration about the mean position O. The motion of the point N along the diameter YY' is simple harmonic.

Hence, the projection of a uniform circular motion on a diameter of a circle is simple harmonic motion.

Displacement in SHM

The distance travelled by the vibrating particle at any instant of time t from its mean position is known as displacement. When the particle is at P, the displacement of the particle along Y axis is y (Fig. 6.3).

Then, in Δ OPN, $\sin\theta = \frac{ON}{OP}$

$$ON = y = OP \sin\theta$$

$$y = OP \sin\omega t \quad (\because \theta = \omega t)$$

since OP = a, the radius of the circle, the displacement of the vibrating particle is

$$y = a \sin\omega t \qquad \ldots(1)$$

The amplitude of the vibrating particle is defined as its maximum displacement from the mean position.

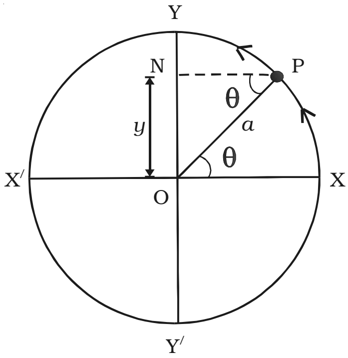


Fig. 6.3 Displacement in SHM

Velocity in SHM

The rate of change of displacement is the velocity of the vibrating particle.

Differentiating eqn. (1) with respect to time t

$$\frac{dy}{dt} = \frac{d}{dt}(a \sin\omega t)$$

$$\therefore v = a\,\omega \cos\omega t \qquad \ldots(2)$$

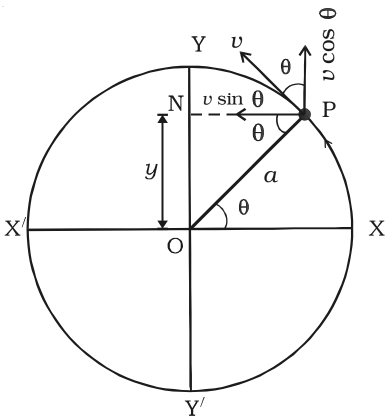


Fig. 6.4 Velocity in SHM

The velocity v of the particle moving along the circle can also be obtained by resolving it into two components as shown in Fig. 6.4.

(i) $v \cos\theta$ in a direction parallel to OY

(ii) $v \sin\theta$ in a direction perpendicular to *OY*

The component $v \sin\theta$ has no effect along *YOY'* since it is perpendicular to *OY*.

$$\therefore \text{Velocity} = v\cos\theta$$

$$= v\cos\omega t$$

We know that, linear velocity = radius × angular velocity

$$\therefore v = a\omega$$

$$\therefore \text{Velocity} = a\omega \cos\omega t$$

$$\therefore \text{Velocity} = a\omega\sqrt{1 - \sin^2\omega t}$$

$$\text{Velocity} = a\omega\sqrt{1-\left(\frac{y}{a}\right)^2} \quad \left[\because \ sin\ \theta = \frac{y}{a}\right]$$

$$\text{Velocity} = \omega\sqrt{a^2 - y^2} \qquad \text{...(3)}$$

Special cases

(i) When the particle is at mean position, (i.e) $y = 0$. Velocity is $a\omega$ and is maximum. $v = \pm a\omega$ is called *velocity amplitude.*

(ii) When the particle is in the extreme position, (i.e) $y = \pm a$, the velocity is zero.

Acceleration in SHM

The rate of change of velocity is the acceleration of the vibrating particle.

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dt}(a\omega \cos \omega t) = -\omega^2\ a\ sin\ \omega t.$$

$$\therefore \text{ acceleration} = \frac{d^2y}{dt^2} = -\omega^2\ y \qquad \text{...(4)}$$

The acceleration of the particle can also be obtained by component method.

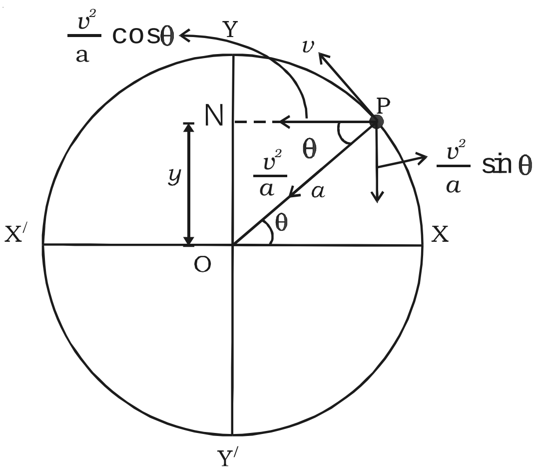


Fig. 6.5 Acceleration in SHM

The centripetal acceleration of the particle P acting along PO is $\frac{v^2}{a}$. This acceleration is resolved into two components as shown in Fig. 6.5.

(i) $\frac{v^2}{a}$ cos θ along PN perpendicular to OY

(ii) $\frac{v^2}{a}$ $sin\ \theta$ in a direction parallal to YO

The component $\frac{v^2}{a}$ *cos* θ has no effect along *YOY′* since it is perpendicular to *OY*.

Hence acceleration $= -\frac{v^2}{a} \sin\theta$

$= -a\,\omega^2 \sin \omega t \qquad (\because v = a\,\omega)$

$= -\omega^2 y \qquad (\because y = a \sin \omega t)$

$\therefore$ acceleation $= -\omega^2 y$

The negative sign indicates that the acceleration is always opposite to the direction of displacement and is directed towards the centre.

Special Cases

(i) When the particle is at the mean position (i.e) $y = 0$, the acceleration is zero.

(ii) When the particle is at the extreme position (i.e) $y = \pm a$, acceleration is $\mp a\,\omega^2$ which is called as *acceleration amplitude.*

The differential equation of simple harmonic motion from eqn. (4) is $\frac{d^2y}{dt^2} + \omega^2 y = 0$ ...(5)

Using the above equations, the values of displacement, velocity and acceleration for the SHM are given in the Table 6.1.

It will be clear from the above, that at the mean position $y = 0$, velocity of the particle is maximum but acceleration is zero. At extreme

Table 6.1 - Displacement, Velocity and Acceleration

Time	ωt	Displacement $a \sin \omega t$	Velocity $a\omega \cos \omega t$	Acceleration $-\omega^2 a \sin \omega t$
$t = 0$	0	0	$a\omega$	0
$t = \frac{T}{4}$	$\frac{\pi}{2}$	$+a$	0	$-a\omega^2$
$t = \frac{T}{2}$	π	0	$-a\omega$	0
$t = \frac{3T}{4}$	$\frac{3\pi}{2}$	$-a$	0	$+a\omega^2$
$t = T$	2π	0	$+a\omega$	0

position $y = \pm a$, the velocity is zero but the acceleration is maximum $\mp a\omega^2$ acting in the opposite direction.

Graphical representation of SHM

Graphical representation of displacement, velocity and acceleration of a particle vibrating simple harmonically with respect to time t is shown in Fig. 6.6.

(i) Displacement graph is a sine curve. Maximum displacement of the particle is $y = \pm a$.

(ii) The velocity of the vibrating particle is maximum at the mean position i.e $v = \pm a\omega$ and it is zero at the extreme position.

(iii) The acceleration of the vibrating particle is zero at the mean position and maximum at the extreme position (i.e) $\mp a\omega^2$.

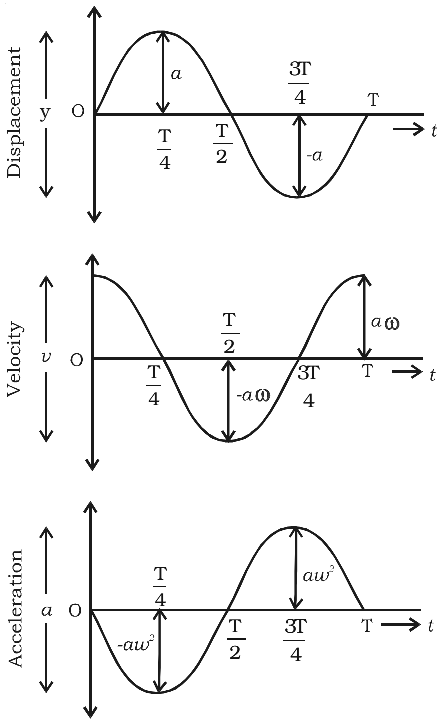


Fig. 6.6 Graphical representation

The velocity is ahead of displacement by a phase angle of $\frac{\pi}{2}$. The acceleration is ahead of the velocity by a phase angle $\frac{\pi}{2}$ or by a phase π ahead of displacement. (i.e) when the displacement has its greatest positive value, acceleration has its negative maximum value or vice versa.

6.2 Important terms in simple harmonic motion

(i) Time period

The time taken by a particle to complete one oscillation is called the time period T.

In the Fig. 6.2, as the particle P completes one revolution with angular velocity ω, the foot of the perpendicular N drawn to the vertical diameter completes one vibration. Hence T is the time period.

Then $\omega = \frac{2\pi}{T}$ or $T = \frac{2\pi}{\omega}$

The displacement of a particle executing simple harmonic motion may be expressed as

$$y(t) = a \sin \frac{2\pi}{T} t \quad ...(1)$$

and $$y(t) = a \cos \frac{2\pi}{T} t \quad ...(2)$$

where T represents the time period, a represents maximum displacement (amplitude).

These functions repeat when t is replaced by $(t + T)$.

$$y(t + T) = a \sin \left[\frac{2\pi}{T} (t + T) \right] \quad ...(3)$$

$$= a \sin \left[2\pi \frac{t}{T} + 2\pi \right]$$

$$= a \sin 2\pi \frac{t}{T} = y(t)$$

In general $y(t + nT) = y(t)$

Above functions are examples of periodic function with time period T. It is clear that the motion repeats after a time $T = \frac{2\pi}{\omega}$ where ω is the angular frequency of the motion. In one revolution, the angle covered by a particle is 2π in time T.

(ii) Frequency and angular frequency

The number of oscillations produced by the body in one second is known as frequency. It is represented by n. The time period to complete one oscillation is $\frac{1}{n}$.

$T = \frac{1}{n}$ shows the time period is the reciprocal of the frequency. Its unit is hertz. $\omega = 2\pi n$, is called as angular frequency. It is expressed in rad s^{-1}.

(iii) Phase

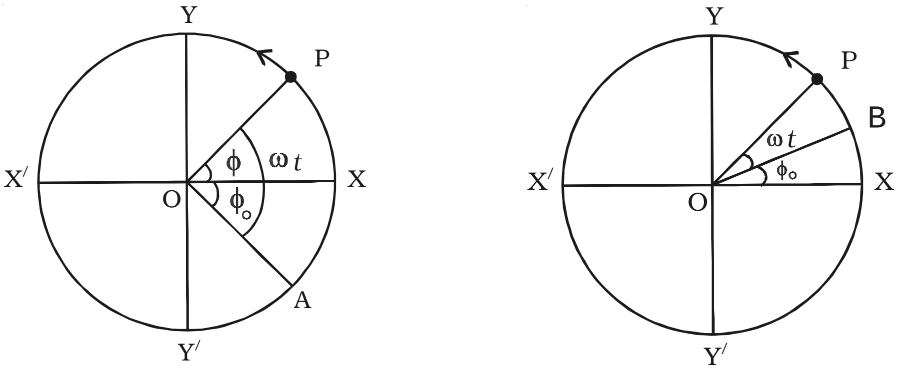
The phase of a particle vibrating in SHM is the state of the particle as regards to its direction of motion and position at any instant of time. In the equation $y = a \sin(\omega t + \phi_o)$ the term $(\omega t + \phi_o) = \phi$, is known as the phase of the vibrating particle.

Epoch

It is the initial phase of the vibrating particle (i.e) phase at t = 0.

$\therefore \ \phi = \phi_o \quad (\because \ \phi = \omega t + \phi_o)$

The phase of a vibrating particle changes with time but the epoch is phase constant.



(a) Phase $\phi = (\omega t - \phi_o)$ (b) Phase $\phi = (\omega t + \phi_o)$

Fig. 6.7 Phase

(i) If the particle *P* starts from the position *X*, the phase of the particle is Zero.

(ii) Instead of counting the time from the instant the particle is at *X*, it is counted from the instant when the reference particle is at *A* (Fig. 6.7a) . Then $\angle XOP = (\omega t - \phi_o)$.

Here $(\omega t - \phi_o) = \phi$ is called the phase of the vibrating particle. $(-\phi_o)$ is initial phase or *epoch*.

(iii) If the time is counted from the instant the particle *P* is above *X* (i.e) at *B*, [Fig. 6.7b] then $(\omega t + \phi_o) = \phi$. Here $(+\phi_o)$ is the *initial phase.*

Phase difference

If two vibrating particles executing SHM with same time period, cross their respective mean positions at the same time in the same direction, they are said to be in phase.

If the two vibrating particles cross their respective mean position at the same time but in the *opposite direction*, they are said to be out of phase (i.e they have a phase difference of π).

If the vibrating motions are represented by equations

$y_1 = a \sin \omega t$ and

$y_2 = a \sin (\omega t - \phi)$

then the phase difference between their phase angles is equal to the phase difference between the two motions.

$\therefore$ phase difference $= \omega t - \phi - \omega t = -\phi$ negative sign indicates that the second motion lags behind the first.

If $y_2 = a \sin (\omega t + \phi)$,

phase difference $= \omega t + \phi - \omega t = \phi$

Here the second motion leads the first motion.

We have discussed the SHM without taking into account the cause of the motion which can be a force (linear SHM) or a torque (angular SHM).

Some examples of SHM

(i) Horizontal and vertical oscillations of a loaded spring.

(ii) Vertical oscillation of water in a U–tube

(iii) Oscillations of a floating cylinder

(iv) Oscillations of a simple pendulum

(v) Vibrations of the prongs of a tuning fork.

6.3 Dynamics of harmonic oscillations

The oscillations of a physical system results from two basic properties namely elasticity and inertia. Let us consider a body displaced from a mean position. The restoring force brings the body to the mean position.

(i) At extreme position when the displacement is maximum, velocity is zero. The acceleration becomes maximum and directed towards the mean position.

(ii) Under the influence of restoring force, the body comes back to the mean position and overshoots because of negative velocity gained at the mean position.

(iii) When the displacement is negative maximum, the velocity becomes zero and the acceleration is maximum in the positive direction. Hence the body moves towards the mean position. Again when the displacement is zero in the mean position velocity becomes positive.

(iv) Due to inertia the body overshoots the mean position once again. This process repeats itself periodically. Hence the system oscillates.

The restoring force is directly proportional to the displacement and directed towards the mean position.

(i.e) $F \; \alpha \; y$

$$F = -ky \qquad \ldots (1)$$

where k is the force constant. It is the force required to give unit displacement. It is expressed in N m^{-1}.

From Newton's second law, $F = ma$ ...(2)

$$\therefore \quad -k\,y = ma$$

$$\text{or} \quad a = -\frac{k}{m}\,y \qquad \ldots(3)$$

From definition of SHM acceleration $a = -\omega^2 y$

The acceleration is directly proportional to the negative of the displacement.

Comparing the above equations we get,

$$\omega = \sqrt{\frac{k}{m}} \qquad \ldots(4)$$

Therefore the period of SHM is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

$$T = 2\pi\sqrt{\frac{\textit{inertial factor}}{\textit{spring factor}}} \qquad \ldots(5)$$

6.4 Angular harmonic oscillator

Simple harmonic motion can also be angular. In this case, the restoring torque required for producing SHM is directly proportional to the angular displacement and is directed towards the mean position.

Consider a wire suspended vertically from a rigid support. Let some weight be suspended from the lower end of the wire. When the wire is twisted through an angle θ from the mean position, a restoring torque acts on it tending to return it to the mean position. Here restoring torque is proportional to angular displacement θ.

Hence $\tau = -C\,\theta$ ...(1)

where C is called torque constant.

It is equal to the moment of the couple required to produce unit angular displacement. Its unit is N m rad^{-1}.

The negative sign shows that torque is acting in the opposite direction to the angular displacement. This is the case of angular simple harmonic motion.

Examples : Torsional pendulum, balance wheel of a watch.

But $\tau = I\,\alpha$ ...(2)

where τ is torque, I is the moment of inertia and α is angular acceleration

∴ Angular acceleration, $\alpha = \frac{\tau}{I} = -\frac{C\theta}{I}$ ...(3)

This is similar to $a = -\omega^2 y$

Replacing y by θ, and a by α we get

$$\alpha = -\omega^2\theta = -\frac{C}{I}\theta$$

$$\therefore \quad \omega = \sqrt{\frac{C}{I}}$$

∴ Period of SHM $T = 2\pi\sqrt{\frac{I}{C}}$

∴ Frequency $n = \frac{1}{T} = \frac{1}{2\pi\sqrt{I/C}} = \frac{1}{2\pi}\sqrt{\frac{C}{I}}$

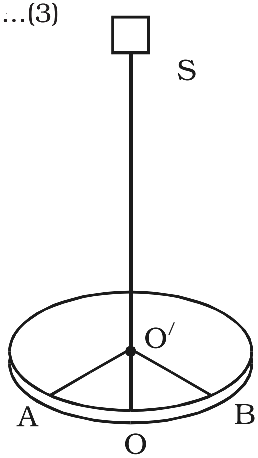


Fig. 6.8 Torsional Pendulum

6.5 Linear simple harmonic oscillator

The block – spring system is a linear simple harmonic oscillator. All oscillating systems like diving board, violin string have some element of springiness, k (spring constant) and some element of inertia, m.

6.5.1 Horizontal oscillations of spring

Consider a mass *(m)* attached to an end of a spiral spring (which obeys Hooke's law) whose other end is fixed to a support as shown in Fig. 6.9. The body is placed on a smooth horizontal surface. Let the body be displaced through a distance x towards right and released. It will oscillate about its mean position. The restoring force acts in the opposite direction and is proportional to the displacement.

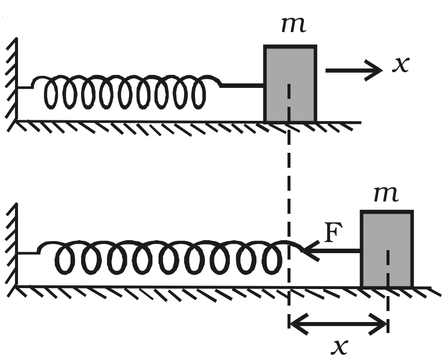


Fig. 6.9 Linear harmonic oscillator

$\therefore$ Restoring force $F = -kx$.

From Newton's second law, we know that $F = ma$

$$\therefore \; ma = -kx$$

$$a = \frac{-k}{m} x$$

Comparing with the equation of SHM $a = -\omega^2 x$, we get

$$\omega^2 = \frac{k}{m}$$

$$\text{or} \quad \omega = \sqrt{\frac{k}{m}}$$

$$\text{But} \quad T = \frac{2\pi}{\omega}$$

$$\text{Time period} \quad T = 2\pi\sqrt{\frac{m}{k}}$$

$$\therefore \text{Frequency} \quad n = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

6.5.2 Vertical oscillations of a spring

Fig 6.10a shows a light, elastic spiral spring suspended vertically from a rigid support in a relaxed position. When a mass '*m*' is attached to the spring as in Fig. 6.10b, the spring is extended by a small length dl such that the upward force F exerted by the spring is equal to the weight mg.

The restoring force $F = k\,\text{dl}$; $k\,dl = mg$ ...(1)

where k is spring constant. If we further extend the given spring by a small distance by applying a small force by our finger, the spring oscillates up and down about its mean position. Now suppose the body is at a distance y above the equilibrium position as in Fig. 6.10c. The extension of the spring is $(dl - y)$. The upward force exerted on the body is $k\,(dl - y)$ and the resultant force F on the body is

$$F = k\,(dl - y) - mg = -ky \qquad ...(2)$$

The resultant force is proportional to the displacement of the body from its equilibrium position and the motion is simple harmonic.

If the total extension produced is $(dl + y)$ as in Fig. 6.10d the restoring force on the body is $k\,(dl + y)$ which acts upwards.

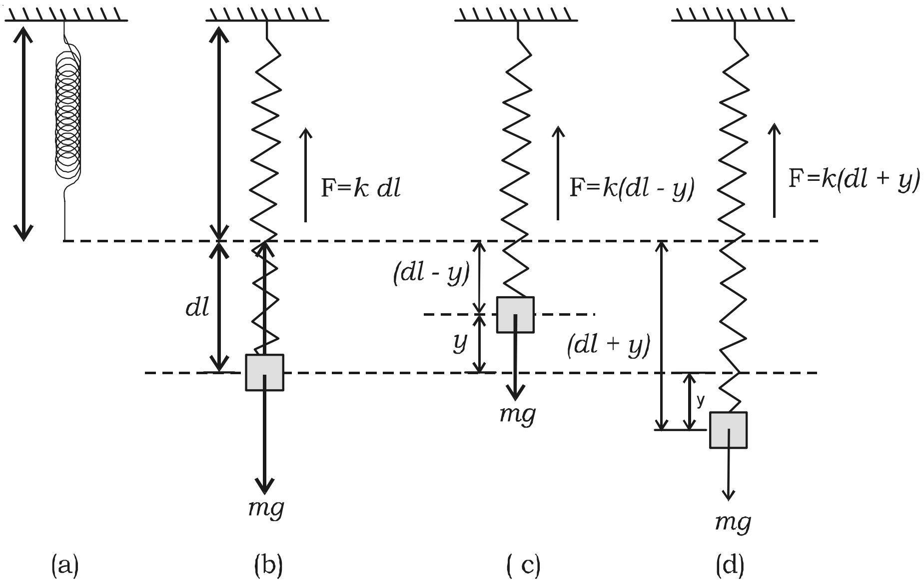


Fig. 6.10 Vertical oscillations of loaded spring

So, the increase in the upward force on the spring is

$$k\,(dl + y) - mg = ky$$

Therefore if we produce an extension downward then the restoring force in the spring increases by ky in the upward direction. As the force acts in the opposite direction to that of displacement, the restoring force is $-ky$ and the motion is SHM.

$$F = -ky$$

$$ma = -ky$$

$$a = -\frac{k}{m}y \quad \text{...(3)}$$

$$a = -\omega^2 y \quad \text{(expression for SHM)}$$

Comparing the above equations, $\omega = \sqrt{\frac{k}{m}}$ ...(4)

But $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$ ...(5)

From equation (1) $mg = k\,dl$

$$\frac{m}{k} = \frac{dl}{g}$$

Therefore time period $T = 2\pi\sqrt{\frac{dl}{g}}$ ...(6)

Frequency $n = \frac{1}{2\pi}\sqrt{\frac{g}{dl}}$

Case 1 : ***When two springs are connected in parallel***

Two springs of spring factors k_1 and k_2 are suspended from a rigid support as shown in Fig. 6.11. A load m is attached to the combination.

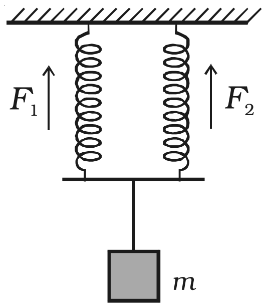


Fig. 6.11 Springs in parallel

Let the load be pulled downwards through a distance y from its equilibrium position. The increase in length is y for both the springs but their restoring forces are different.

If F_1 and F_2 are the restoring forces

$$F_1 = -k_1 y, \qquad F_2 = -k_2 y$$

$\therefore$ Total restoring force = $(F_1 + F_2) = -(k_1 + k_2)\, y$

So, time period of the body is given by

$$T = 2\pi \sqrt{\frac{m}{k_1 + k_2}}$$

If $k_1 = k_2 = k$

Then, $$T = 2\pi \sqrt{\frac{m}{2k}}$$

$\therefore$ frequency $$n = \frac{1}{2\pi} \sqrt{\frac{2k}{m}}$$

Case 2 : *When two springs are connected in series.*

Two springs are connected in series in two different ways. This arrangement is shown in Fig. 6.12a and 6.12b.

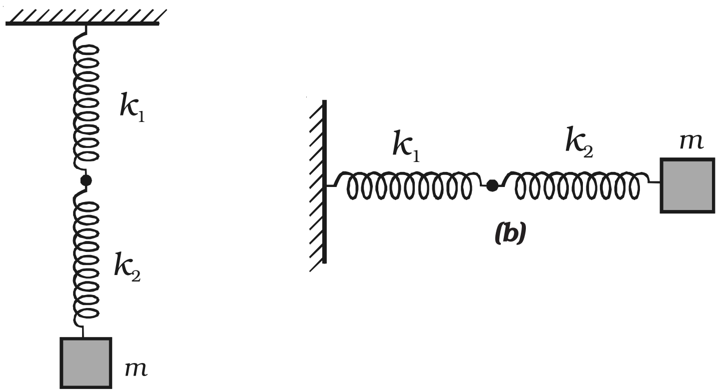


Fig. 6.12 Springs in series

In this system when the combination of two springs is displaced to a distance y, it produces extension y_1 and y_2 in two springs of force constants k_1 and k_2.

$$F = -k_1\, y_1 \quad ; \quad F = -k_2\, y_2$$

where F is the restoring force.

Total extension, $y = y_1 + y_2 = -F\left[\frac{1}{k_1}+\frac{1}{k_2}\right]$

We know that $F = -ky$

$$\therefore\ y = -\frac{F}{k}$$

From the above equations,

$$-\frac{F}{k} = -F\left[\frac{1}{k_1}+\frac{1}{k_2}\right]$$

or $$k = \frac{k_1 k_2}{k_1+k_2}$$

$$\therefore \text{ Time period } = T = 2\pi\sqrt{\frac{m(k_1+k_2)}{k_1 k_2}}$$

$$\text{frequency } n = \frac{1}{2\pi}\sqrt{\frac{k_1 k_2}{(k_1+k_2)m}}$$

If both the springs have the same spring constant,

$$k_1 = k_2 = k.$$

$$\therefore\ n = \frac{1}{2\pi}\sqrt{\frac{k}{2m}}$$

6.5.3 Oscillation of liquid column in a U - tube

Consider a non viscous liquid column of length l of uniform cross-sectional area A (Fig. 6.13). Initially the level of liquid in the limbs is the same. If the liquid on one side of the tube is depressed by blowing gently the levels of the liquid oscillates for a short time about their initial positions O and C, before coming to rest.

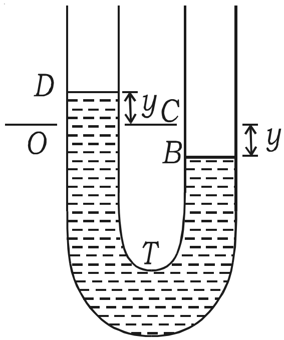


Fig. 6.13 Oscillation of a liquid column in U - tube

If the liquid in one of the limbs is depressed by y , there will be a difference of 2 y in the liquid levels in the two limbs. At some instant, suppose the level of the liquid on the left side of the tube is at D, at a height y above its original position O, the level B of the liquid on the other side is then at a depth y below its original position C. So the excess pressure P on the liquid due to the restoring force is excess height × density × g

(i.e) pressure = $2\, y\, \rho\, g$

$\therefore$ Force on the liquid = pressure × area of the cross-section of the tube

$$= -2\, y\, \rho\, g \times A \qquad \text{..... (1)}$$

The negative sign indicates that the force towards O is opposite to the displacement measured from O at that instant.

The mass of the liquid column of length l is volume × density

(i.e) $m = l\, A\, \rho$

$$\therefore F = l\, A\, \rho\, a \qquad \text{.... (2)}$$

From equations (1) and (2) $\quad l\, A\, \rho\, a = -\, 2\, y\, A\, \rho\, g$

$$\therefore a = -\frac{2g}{l}\, y \qquad \text{..... (3)}$$

We know that $a = -\omega^2 y$

(i.e) $a = -\dfrac{2g}{l}\, y = -\omega^2 y \qquad$ where $\quad \omega = \sqrt{\dfrac{2g}{l}}$

Here, the acceleration is proportional to the displacement, so the motion is simple harmonic and the period T is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{2g}}$$

6.5.4 *Oscillations of a simple pendulum*

A simple pendulum consists of massless and inelastic thread whose one end is fixed to a rigid support and a small bob of mass m is suspended from the other end of the thread. Let l be the length of the pendulum. When the bob is slightly displaced and released, it oscillates about its equilibrium position. Fig.6.14 shows the displaced position of the pendulum.

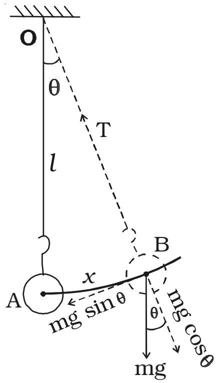


Fig. 6.14
Simple Pendulum – Linear SHM

Suppose the thread makes an angle θ with the vertical. The distance of the bob from the equilibrium position *A* is *AB*. At *B*, the weight *mg* acts vertically downwards. This force is resolved into two components.

(i) The component *mg cos* θ is balanced by the tension in the thread acting along the length towards the fixed point O.

(ii) *mg sin* θ which is unbalanced, acts perpendicular to the length of thread. This force tends to restore the bob to the mean position. If the amplitude of oscillation is small, then the path of the bob is a straight line.

$$\therefore \quad F = -mg \sin\theta \qquad ...(1)$$

If the angular displacement is small sin θ ≈ θ

$$\therefore \ F = -mg\,\theta \qquad ...(2)$$

But $\theta = \dfrac{x}{l}$

$$\therefore \ F = -\,mg\,\frac{x}{l}$$

Comparing this equation with Newton's second law, $F = ma$ we get, acceleration $a = -\dfrac{gx}{l}$ ...(3)

(negative sign indicates that the direction of acceleration is opposite to the displacement) Hence the motion of simple pendulum is SHM.

We know that $a = -\omega^2 x$

Comparing this with (3)

$$\omega^2 = \frac{g}{l} \text{ or } \omega = \sqrt{\frac{g}{l}} \qquad ...(4)$$

$$\therefore \quad \text{Time period } T = \frac{2\pi}{\omega}$$

$$T = 2\pi\sqrt{\frac{l}{g}} \qquad ...(5)$$

$$\therefore \quad \text{frequency } n = \frac{1}{2\pi}\sqrt{\frac{g}{l}} \qquad ...(6)$$

Laws of pendulum

From the expression for the time period of oscillations of a pendulum the following laws are enunciated.

(i) The law of length

The period of a simple pendulum varies directly as the square root of the length of the pendulum.

(i.e) $T \alpha \sqrt{l}$

(ii) The law of acceleration

The period of a simple pendulum varies inversely as the square root of the acceleration due to gravity.

(i.e) $T \alpha \frac{1}{\sqrt{g}}$

(iii) The law of mass

The time period of a simple pendulum is independent of the mass and material of the bob.

(iv) The law of amplitude

The period of a simple pendulum is independent of the amplitude provided the amplitude is small.

Note : The length of a seconds pendulum is 0.99 m whose period is two seconds.

$$2 = 2\pi \sqrt{\frac{l}{g}}$$

$$\therefore \; l = \frac{9.81 \times 4}{4\pi^2} = 0.99 \text{ m}$$

Oscillations of simple pendulum can also be regarded as a case of *angular SHM.*

Let θ be the angular displacement of the bob B at an instant of time. The bob makes rotation about the horizontal line which is perpendicular to the plane of motion as shown in Fig. 6.15.

Restoring torque about O is $\tau = -\, mg\, l \sin \theta$

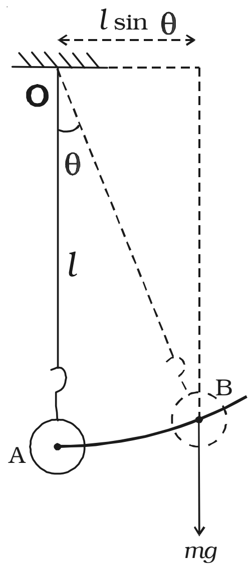


Fig. 6.15 Simple pendulum - Angular SHM

$$\tau = -m\,g\,l\,\theta \quad [\because \theta \text{ is small}] \qquad ...(1)$$

Moment of inertia
about the axis = $m\,l^2$ ...(2)

If the amplitude is small, motion of the bob is angular simple harmonic. Therefore angular acceleration of the system about the axis of rotation is

$$\alpha = \frac{\tau}{I} = \frac{-mgl\theta}{ml^2}$$

$$\alpha = -\frac{g}{l}\theta \qquad ...(3)$$

We know that $\alpha = -\omega^2 \theta$ ...(4)

Comparing (3) and (4)

$$-\omega^2\theta = -\frac{g}{l}\theta$$

angular frequency $\omega = \sqrt{\frac{g}{l}}$

Time period $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}$ ...(5)

Frequency $n = \frac{1}{2\pi}\sqrt{\frac{g}{l}}$ ...(6)

6.6 Energy in simple harmonic motion

The total energy (E) of an oscillating particle is equal to the sum of its kinetic energy and potential energy if conservative force acts on it.

The velocity of a particle executing SHM at a position where its displacement is y from its mean position is $v = \omega\sqrt{a^2 - y^2}$

Kinetic energy

Kinetic energy of the particle of mass m is

$$K = \frac{1}{2}m\left[\omega\sqrt{a^2 - y^2}\right]^2$$

$$K = \frac{1}{2}\, m\, \omega^2\, (a^2 - y^2) \qquad ...(1)$$

Potential energy

From definition of SHM $F = -ky$ the work done by the force during the small displacement dy is $dW = -F.dy = -(-ky)\ dy = ky\ dy$

$\therefore$ Total work done for the displacement y is,

$$W = \int dW = \int_0^y ky\ dy$$

$$W = \int_0^y m\omega^2 y\, dy \quad [\because k = m\omega^2)$$

$$\therefore W = \frac{1}{2} m\,\omega^2 y^2$$

This work done is stored in the body as potential energy

$$U = \frac{1}{2} m\,\omega^2 y^2 \qquad \text{...(2)}$$

Total energy $E = K + U$

$$= \frac{1}{2} m\omega^2 (a^2 - y^2) + \frac{1}{2} m\,\omega^2 y^2$$

$$= \frac{1}{2} m\,\omega^2 a^2$$

Thus we find that the total energy of a particle executing simple harmonic motion is $\frac{1}{2} m\,\omega^2 a^2$.

Special cases

(i) When the particle is at the mean position $y = 0$, from eqn (1) it is known that kinetic energy is maximum and from eqn. (2) it is known that potential energy is zero. Hence the total energy is wholly kinetic

$$E = K_{max} = \frac{1}{2} m\omega^2 a^2$$

(ii) When the particle is at the extreme position y = $\pm a$, from eqn. (1) it is known that kinetic energy is zero and from eqn. (2) it is known that Potential energy is maximum. Hence the total energy is wholly potential.

$$E = U_{max} = \frac{1}{2} m \omega^2 a^2$$

(iii) when $y = \frac{a}{2}$,

$$K = \frac{1}{2} m \omega^2 \left[a^2 - \frac{a^2}{4}\right]$$

$$\therefore K = \frac{3}{4}\left(\frac{1}{2} m\omega^2 a^2\right)$$

$$K = \frac{3}{4} E$$

$$U = \frac{1}{2} m \omega^2 \left(\frac{a}{2}\right)^2 = \frac{1}{4}\left(\frac{1}{2} m \omega^2 a^2\right)$$

$$\therefore \quad U = \frac{1}{4} E$$

If the displacement is half of the amplitude, $K = \frac{3}{4} E$ and $U = \frac{1}{4} E$. K and U are in the ratio 3 : 1,

$$E = K + U = \frac{1}{2} m \omega^2 a^2$$

At any other position the energy is partly kinetic and partly potential.

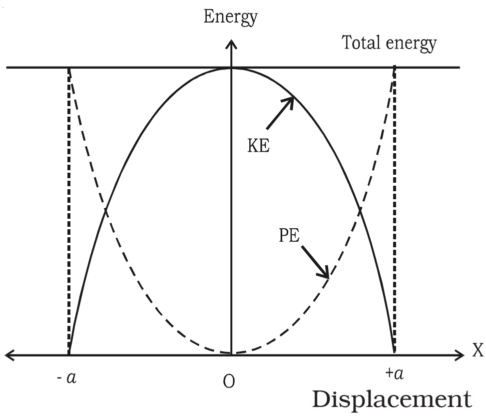


Fig. 6.16 Energy – displacement graph

This shows that the particle executing SHM obeys the law of conservation of energy.

Graphical representation of energy

The values of K and U in terms of E for different values of y are given in the Table 6.2. The variation of energy of an oscillating particle with the displacement can be represented in a graph as shown in the Fig. 6.16.

Table 6.2 ***Energy of SHM***

y	0	$\frac{a}{2}$	a	$-\frac{a}{2}$	$-a$
Kinetic energy	E	$\frac{3}{4}$ E	0	$\frac{3}{4}$ E	0
Potential energy	0	$\frac{1}{4}$ E	E	$\frac{1}{4}$ E	E

6.7 Types of oscillations

There are three main types of oscillations.

(i) Free oscillations

When a body vibrates with its own natural frequency, it is said to execute free oscillations. The frequency of oscillations depends on the inertial factor and spring factor, which is given by,

$$n = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Examples

(i) Vibrations of tuning fork

(ii) Vibrations in a stretched string

(iii) Oscillations of simple pendulum

(iv) Air blown gently across the mouth of a bottle.

(ii) Damped oscillations

Most of the oscillations in air or in any medium are damped. When an oscillation occurs, some kind of damping force may arise due to friction or air resistance offered by the medium. So, a part of the energy is dissipated in overcoming the resistive force. Consequently, the amplitude of oscillation decreases with time and finally becomes zero. Such oscillations are called damped oscillations (Fig. 6.17).

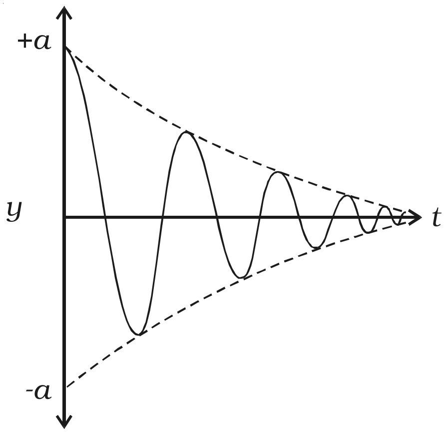


Fig. 6.17 Damped oscillations

Examples :

(i) The oscillations of a pendulum

(ii) Electromagnetic damping in galvanometer (oscillations of a coil in galvanometer)

(iii) Electromagnetic oscillations in tank circuit

(iii) Maintained oscillations

The amplitude of an oscillating system can be made constant by feeding some energy to the system. If an energy is fed to the system to compensate the energy it has lost, the amplitude will be a constant. Such oscillations are called *maintained oscillations* (Fig. 6.18).

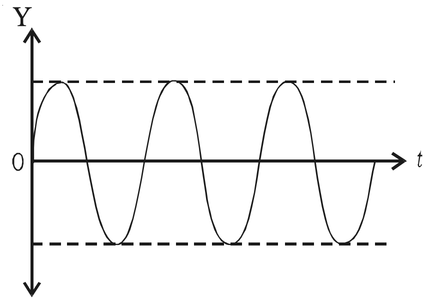


Fig. 6.18 Maintained oscillations

Example :

A swing to which energy is fed continuously to maintain amplitude of oscillation.

(iv) Forced oscillations

When a vibrating body is maintained in the state of vibration by a periodic force of frequency (n) other than its natural frequency of the body, the vibrations are called forced vibrations. The external force is *driver* and body is *driven.*

The body is forced to vibrate with an external periodic force. The amplitude of forced vibration is determined by the difference between the frequencies of the driver and the driven. The larger the frequency difference, smaller will be the amplitude of the forced oscillations.

Examples :

(i) Sound boards of stringed instruments execute forced vibration,

(ii) Press the stem of vibrating tuning fork, against tabla. The tabla suffers forced vibration.

(v) Resonance

In the case of forced vibration, if the frequency difference is small, the amplitude will be large (Fig. 6.19). Ultimately when the two frequencies are same, amplitude becomes maximum.This is a *special case of forced vibration.*

If the frequency of the external periodic force is equal to the natural frequency of oscillation of the system, then the amplitude of oscillation will be large and this is known as resonance.

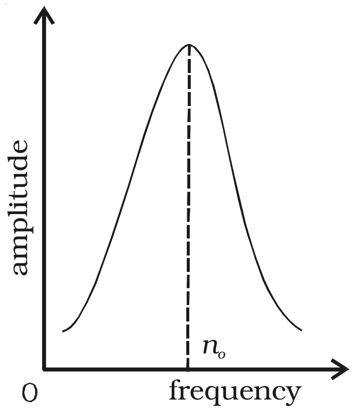


Fig. 6.19 Resonance

Advantages

(i) Using resonance, frequency of a given tuning fork is determined with a sonometer.

(ii) In radio and television, using tank circuit, required frequency can be obtained.

Disadvantages

(i) Resonance can cause disaster in an earthquake, if the natural frequency of the building matches the frequency of the periodic oscillations present in the Earth. The building begins to oscillate with large amplitude thus leading to a collapse.

(ii) A singer maintaining a note at a resonant frequency of a glass, can cause it to shatter into pieces.

Solved problems

6.1 Obtain an equation for the SHM of a particle whose amplitude is 0.05 m and frequency 25 Hz. The initial phase is $\pi/3$.

Data : $a = 0.05\ m,\ n = 25\ Hz,\ \phi_o = \pi/3.$

Solution : $\omega = 2\pi n = 2\pi \times 25 = 50\pi$

The equation of SHM is $y = a \sin(\omega t + \phi_o)$

The displacement equation of SHM is : $y = 0.05 \sin(50\pi t + \pi/3)$

6.2 The equation of a particle executing SHM is $y = 5 \sin\left(\pi t + \frac{\pi}{3}\right)$.

Calculate (i) amplitude (ii) period (iii) maximum velocity and (iv) velocity after 1 second (y is in metre).

Data : $y = 5 \sin\left(\pi t + \frac{\pi}{3}\right)$

Solution : *The equation of SHM is* $y = a \sin(\omega t + \phi_o)$

Comparing the equations

(i) Amplitude $a = 5\ m$

(ii) Period, $T = \frac{2\pi}{\omega} = \frac{2\pi}{\pi} = 2\ s$

(iii) $v_{max} = a\omega = 5 \times \pi = 15.7\ m\ s^{-1}$

(iv) Velocity after 1 s $= a\omega \cos(\omega t + \phi_o)$

$$= 15.7\left[\cos\left(\pi \times 1 + \frac{\pi}{3}\right)\right]$$

$$= 15.7 \times \frac{1}{2} = 7.85\ m\ s^{-1}$$

$$\therefore v = 7.85\ m\ s^{-1}$$

6.3 A particle executes a simple harmonic motion of time period T. Find the time taken by the particle to have a displacement from mean position equal to one half of the amplitude.

Solution : *The displacement is given by* $y = a \sin \omega t$

When the displacement $y = \frac{a}{2}$,

we get $$\frac{a}{2} = a \sin \omega t$$

or $$\sin \omega t = \frac{1}{2}$$

$$\omega t = \frac{\pi}{6}$$

$$\therefore \quad t = \frac{\pi}{6\omega} = \frac{\pi}{6.\frac{2\pi}{T}}$$

The time taken is $t = \frac{T}{12}$ *s*

6.4 The velocities of a particle executing SHM are 4 cm s^{-1} and 3 cm s^{-1}, when its distance from the mean position is 2 cm and 3 cm respectively. Calculate its amplitude and time period.

Data : $v_1 = 4$ cm $s^{-1} = 4 \times 10^{-2}$ m s^{-1} ; $v_2 = 3$ cm $s^{-1} = 3 \times 10^{-2}$ m s^{-1}

$y_1 = 2$ cm $= 2 \times 10^{-2}$ m ; $y_2 = 3$ cm $= 3 \times 10^{-2}$ m

Solution : $$v_1 = \omega \sqrt{a^2 - y_1^2} \quad \ldots (1)$$

$$v_2 = \omega \sqrt{a^2 - y_2^2} \quad \ldots (2)$$

Squaring and dividing the equations

$$\left(\frac{v_1}{v_2}\right)^2 = \frac{a^2 - y_1^2}{a^2 - y_2^2}$$

$$\left(\frac{4 \times 10^{-2}}{3 \times 10^{-2}}\right)^2 = \frac{a^2 - 4 \times 10^{-4}}{a^2 - 9 \times 10^{-4}}$$

$$9a^2 - 36 \times 10^{-4} = 16a^2 - 144 \times 10^{-4}$$

$$7a^2 = 108 \times 10^{-4}$$

$$\therefore a = \sqrt{15.42} \times 10^{-2} = 0.03928 \text{ m}$$

Substituting the value of a^2 *in equation (1)*

we have

$$4 \times 10^{-2} = \omega \sqrt{\frac{108\times10^{-4}}{7} - 4\times10^{-4}}$$

$$\therefore \quad \omega = \sqrt{\frac{7}{5}} \ rad\ s^{-1}$$

$$\therefore \quad \text{Time period } T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{5}{7}}$$

$$T = 5.31\ s$$

6.5 A circular disc of mass 10 kg is suspended by a wire attached to its centre. The wire is twisted by rotating the disc and released. The period of torsional oscillation is found to be 1.5 s. The radius of the disc is 15 cm. Calculate the torsional spring constant.

Data : $m = 10\ kg,\ T = 1.5\ s,\ r = 15\ cm = 15 \times 10^{-2}\ m \quad C = ?$

Solution : *MI of the disc about an axis through the centre is*

$$I = \frac{1}{2} MR^2$$

The time period of angular SHM is $T = 2\pi\sqrt{\frac{I}{C}}$

Squaring the equation, $T^2 = 4\pi^2 \frac{I}{C}$

$$\therefore \quad C = \frac{4\pi^2 I}{T^2}$$

$$C = \frac{4\pi^2 \times \frac{1}{2} MR^2}{T^2}$$

$$= \frac{2\times(3.14)^2\times10\times0.15^2}{(1.5)^2}$$

$$C = 2.0\ N\ m\ rad^{-1}$$

6.6 A body of mass 2 kg executing SHM has a displacement $y = 3\ sin\left(100\ t + \frac{\pi}{4}\right)$ cm. Calculate the maximum kinetic energy of the body.

Solution : *Comparing with equation of SHM*

$$y = a \sin(\omega t + \phi_o)$$

$$a = 3\ cm = 3 \times 10^{-2}\ m,\ \omega = 100\ rad\ s^{-1},\ m = 2\ kg$$

$$y = 3 \sin\left(100\ t + \frac{\pi}{4}\right)$$

$$\text{Maximum kinetic energy} = \frac{1}{2}\ ma^2\ \omega^2$$

$$= \frac{1}{2} \times 2 \times (0.03^2 \times 100^2)$$

$$\text{Maximum kinetic energy} = 9\ \text{joule}$$

6.7 A block of mass 15 kg executes SHM under the restoring force of a spring. The amplitude and the time period of the motion are 0.1 m and 3.14 s respectively. Find the maximum force exerted by the spring on the block.

Data : *$m = 15$ kg, $a = 0.1$ m and $T = 3.14$ s*

Solution : *The maximum force exerted on the block is ka, when the block is at the extreme position, where k is the spring constant.*

$$\text{The angular frequency} = \omega = \frac{2\pi}{T} = 2\ s^{-1}$$

$$\text{The spring constant } k = m\,\omega^2$$

$$= 15 \times 4 = 60\ N\ m^{-1}$$

The maximum force exerted on the block is $ka = 60 \times 0.1 = 6$ N

6.8 A block of mass 680 g is attached to a horizontal spring whose spring constant is 65 Nm^{-1}. The block is pulled to a distance of 11 cm from the mean position and released from rest. Calculate : (i) angular frequency, frequency and time period (ii) displacement of the system (iii) maximum speed and acceleration of the system

Data : *$m = 680\ g = 0.68\ kg$, $k = 65\ N\ m^{-1}$, $a = 11\ cm = 0.11\ m$*

Solution : *The angular frequency* $\omega = \sqrt{\frac{k}{m}}$

$$\omega = \sqrt{\frac{65}{0.68}} = 9.78\ rad\ s^{-1}$$

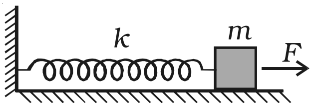
The frequency $n = \frac{\omega}{2\pi} = \frac{9.78}{2\pi} = 1.56\ Hz$

The time period $T = \frac{1}{n} = \frac{1}{1.56} = 0.64\ s$

maximum speed $= a\,\omega$

$= 0.11 \times 9.78$

$= 1.075\ m\ s^{-1}$



Acceleration of the block $= a\,\omega^2 = a\omega \times \omega$

$= 1.075 \times (9.78)$

$= 10.52\ m\ s^{-2}$

Displacement $y(t) = a \sin \omega t$

$\therefore\ y(t) = 0.11 \sin 9.78\, t$ *metre*

6.9 A mass of 10 kg is suspended by a spring of length 60 cm and force constant 4×10^3 N m^{-1}. If it is set into vertical oscillations, calculate the (i) frequency of oscillation of the spring and (ii) the length of the stretched string.

Data : $k = 4 \times 10^3\ N\,m^{-1}$, $F = 10 \times 9.8\ N$, $l = 60 \times 10^{-2}\ m$, $m = 10\ kg$

Solution : (i) $n = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$

$$= \frac{1}{2\pi}\sqrt{\frac{4 \times 10^3}{10}} = \frac{20}{2\pi}$$

Frequency $= 3.184\ Hz$

(ii) $T = 2\pi\sqrt{\frac{dl}{g}}$ or $T^2 = 4\pi^2 \frac{dl}{g}$

length $(dl) = \frac{T^2 g}{4\pi^2} = \frac{1}{n^2} \times \frac{g}{4\pi^2}$

$$\therefore\quad dl = \frac{9.8}{(3.184)^2 \times 4 \times (3.14)^2}$$

$dl = 0.0245\ m$

$\therefore$ The length of the stretched string = 0.6 + 0.0245 = 0.6245 m

6.10 A mass m attached to a spring oscillates every 4 seconds. If the mass is increased by 4 kg, the period increases by 1 s. Find its initial mass m.

Data : *Mass m oscillates with a period of 4 s*

When the mass is increased by 4 kg period is 5 s

Solution : *Period of oscillation* $T = 2\pi \sqrt{\frac{m}{k}}$

$$4 = 2\pi \sqrt{m/k} \qquad \dots (1)$$

$$5 = 2\pi \sqrt{\frac{m+4}{k}} \qquad \dots (2)$$

Squaring and dividing the equations

$$\frac{25}{16} = \frac{m+4}{m}$$

$$25\,m = 16\,m + 64$$

$$9m = 64$$

$$\therefore \quad m = \frac{64}{9} = 7.1\ kg$$

6.11 The acceleration due to gravity on the surface of moon is 1.7 m s^{-2}. What is the time period of a simple pendulum on the surface of the moon, if its period on the Earth is 3.5 s ?

Data : g on moon = 1.7 m s^{-2}

g on the Earth = 9.8 ms^{-2}

Time period on the Earth = 3.5 s

Solution : $T = 2\pi \sqrt{\frac{l}{g}}$

Let T_m represent the time period on moon

$$T_m = 2\pi \sqrt{\frac{l}{1.7}} \qquad \dots (1)$$

On the Earth, $3.5 = 2\pi \sqrt{\frac{l}{9.8}}$ ... (2)

Dividing the equation (2) by (1) and squaring

$$\left(\frac{3.5}{T_m}\right)^2 = \frac{1.7}{9.8}$$

$$T_m{}^2 \times 1.7 = (3.5)^2 \times 9.8$$

$$T_m{}^2 = \frac{(3.5)^2 \times 9.8}{1.7} = \frac{12.25 \times 9.8}{1.7}$$

$$\therefore \quad T_m = \sqrt{\frac{120.05}{1.7}} = 8.40\ s$$

6.12 A simple pendulum has a period 4.2 s. When the pendulum is shortened by 1 m the period is 3.7 s. Calculate its (i) acceleration due to gravity (ii) original length of the pendulum.

Data : *T = 4.2 s ; when length is shortened by 1m the period is 3.7 s.*

Solution : $T = 2\pi\sqrt{\frac{l}{g}}$

Squaring and rearranging $g = 4\pi^2 \frac{l}{T^2}$

$$g = 4\pi^2 \frac{l}{(4.2)^2} \quad \ldots(1)$$

When the length is shortened by 1 m

$$g = \frac{4\pi^2(l-1)}{(3.7)^2} \quad \ldots(2)$$

From the above equations

$$\frac{l}{(4.2)^2} = \frac{l-1}{(3.7)^2}$$

$$(7.9 \times 0.5)\, l = 17.64$$

$$l = \frac{17.64}{7.9 \times 0.5} = 4.46\ m$$

Substituting in equation (1)

$$g = 4\pi^2 \frac{4.46}{(4.2)^2} = \frac{175.89}{17.64}$$

$$g = 9.97\ m\ s^{-2}$$

7. Wave Motion

Wave motion is a mode of transmission of energy through a medium in the form of a disturbance. It is due to the repeated periodic motion of the particles of the medium about an equilibrium position transferring the energy from one particle to another.

The waves are of three types - mechanical, electromagnetic and matter waves. Mechanical waves can be produced only in media which possess elasticity and inertia. Water waves, sound waves and seismic waves are common examples of this type. Electromagnetic waves do not require any material medium for propagation. Radio waves, microwaves, infrared rays, visible light, the ultraviolet rays, X rays and γ rays are electromagnetic waves. The waves associated with particles like electrons, protons and fundamental particles in motion are matter waves.

Waves on surface of water

In order to understand the concept of wave motion, let us drop a stone in a trough of water. We find that small circular waves seem to originate from the point where the stone touches the surface of water. These waves spread out in all directions. It appears as if water moves away from that point. If a piece of paper is placed on the water surface, it will be observed that the piece of paper moves up and down, when the waves pass through it. This shows that the waves are formed due to the vibratory motion of the water particles, about their mean position.

Wave motion is a form of disturbance which travels through a medium due to the repeated periodic motion of the particles of the medium about their mean position. The motion is transferred continuously from one particle to its neighbouring particle.

7.1 Characteristics of wave motion

(i) Wave motion is a form of disturbance travelling in the medium due to the periodic motion of the particles about their mean position.

(ii) It is necessary that the medium should possess elasticity and inertia.

(iii) All the particles of the medium do not receive the disturbance at the same instant (i.e) each particle begins to vibrate a little later than its predecessor.

(iv) The wave velocity is different from the particle velocity. The velocity of a wave is constant for a given medium, whereas the velocity of the particles goes on changing and it becomes maximum in their mean position and zero in their extreme positions.

(v) During the propagation of wave motion, there is transfer of energy from one particle to another without any actual transfer of the particles of the medium.

(vi) The waves undergo reflection, refraction, diffraction and interference.

7.1.1 Mechanical wave motion

The two types of mechanical wave motion are (i) transverse wave motion and (ii) longitudinal wave motion

(i) Transverse wave motion

Transverse wave motion is that wave motion in which particles of the medium execute SHM about their mean positions in a direction perpendicular to the direction of propagation of the wave. Such waves are called transverse waves. Examples of transverse waves are waves produced by plucked strings of veena, sitar or violin and electromagnetic waves. Transverse waves travel in the form of crests and troughs. *The maximum displacement of the particle in the positive direction i.e. above its mean position is called crest and maximum displacement of the particle in the negative direction i.e below its mean position is called trough.*

Thus if ABCDEFG is a transverse wave, the points B and F are crests while D is trough (Fig. 7.1).

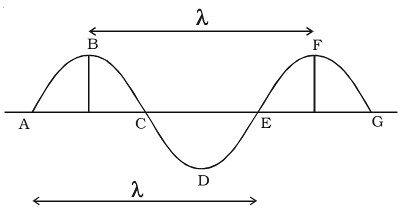


Fig. 7.1 Transverse wave

For the propagation of transverse waves, the medium must possess force of cohesion and volume elasticity. Since gases and liquids do not have rigidity (cohesion), transverse waves cannot be produced in gases and liquids. Transverse waves can be produced in solids and surfaces of liquids only.

(ii) Longitudinal wave motion

'*Longitudinal wave motion is that wave motion in which each particle of the medium executes simple harmonic motion about its mean position along the direction of propagation of the wave.*'

Sound waves in fluids (liquids and gases) are examples of longitudinal wave. When a longitudinal wave travels through a medium, it produces compressions and rarefactions.

In the case of a spiral spring, whose one end is tied to a hook of a wall and the other end is moved forward and backward, the coils of the spring vibrate about their original position along the length of the spring and longitudinal waves propagate through the spring (Fig.7.2).

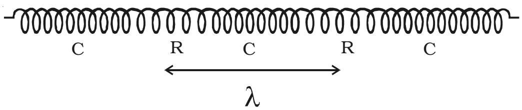


Fig. 7.2 Compression and rarefaction in spring

The regions where the coils are closer are said to be in the state of compression, while the regions where the coils are farther are said to be in the state of rarefaction.

When we strike a tuning fork on a rubber pad, the prongs of the tuning fork begin to vibrate to and fro about their mean positions. When the prong A moves outwards to A_1, it compresses the layer of air in its neighbourhood. As the compressed layer moves forward it compresses the next layer and a wave of compression passes through air. But when the prong moves inwards to A_2, the particles of the medium which moved to the right, now move backward to the left due to elasticity of air. This gives rise to rarefaction.

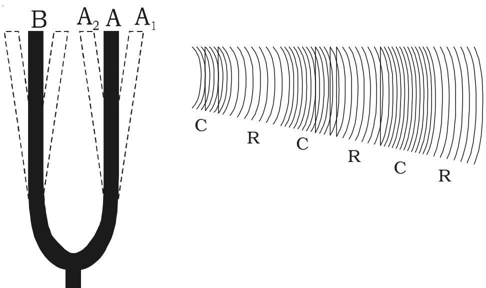


Fig. 7.3 Longitudinal wave

Thus a longitudinal wave is characterised by the formation of compressions and rarefactions following each other.

Longitudinal waves can be produced in all types of material medium, solids, liquids and gases. The density and pressure of the medium in the region of compression are more than that in the region of rarefaction.

7.1.2 Important terms used in wave motion

(i) Wavelength (λ)

The distance travelled by a wave during which a particle of the medium completes one vibration is called wavelength. It is also defined as the distance between any two nearest particles on the wave having same phase.

Wavelength may also be defined as the distance between two successive crests or troughs in transverse waves, or the distance between two successive compressions or rarefactions in longitudinal waves.

(ii) Time period (T)

The time period of a wave is the time taken by the wave to travel a distance equal to its wavelength.

(iii) Frequency (n)

This is defined as the number of waves produced in one second. If T represents the time required by a particle to complete one vibration, then it makes $\frac{1}{T}$ waves in one second.

Therefore frequency is the reciprocal of the time period (i.e) $n = \frac{1}{T}$.

Relationship between velocity, frequency and wavelength of a wave

The distance travelled by a wave in a medium in one second is called the velocity of propagation of the wave in that medium. If v represents the velocity of propagation of the wave, it is given by

$$\text{Velocity} = \frac{\text{Distance travelled}}{\text{Time taken}}$$

$$v = \frac{\lambda}{T} = n\lambda \qquad \left[\because n = \frac{1}{T}\right]$$

The velocity of a wave (v) is given by the product of the frequency and wavelength.

7.2 *Velocity of wave in different media*

The velocity of mechanical wave depends on elasticity and inertia of the medium.

7.2.1 Velocity of a transverse wave along a stretched string

Let us consider a string fixed at one of its ends and tension be applied at the other end. When the string is plucked at a point, it begins to vibrate.

Consider a transverse wave proceeding from left to right in the form of a pulse when the string is plucked at a point as shown in Fig. 7.4. *EF* is the displaced position of the string at an instant of time. It forms an arc of a circle with *O* as centre and R as radius. The arc *EF* subtends an angle 2θ at *O*.

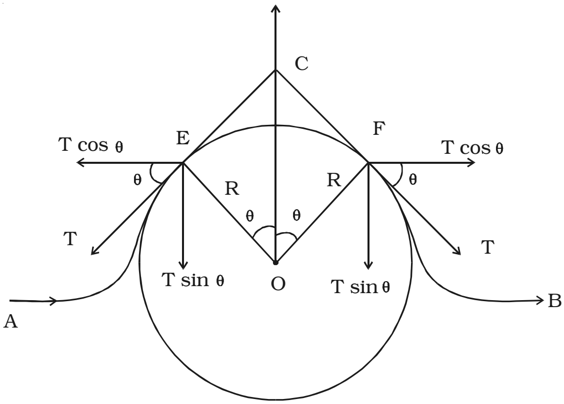


Fig. 7.4 Transverse vibration of a string

If *m* is the mass per unit length of the string and *dx* is the length of the arc *EF*, then the mass of the portion of the string is *m dx*.

$$\therefore \text{ Centripetal force } = \frac{m.dx.v^2}{R} \qquad \ldots(1)$$

This force is along CO. To find the resultant of the tension T at the points E and F, we resolve T into two components Tcos θ and T sinθ.

T cosθ components acting perpendi- cular to CO are of equal in magnitude but opposite in direction, they cancel each other.

T sin θ components act parallel to CO. Therefore the resultant of the tensions acting at E and F is 2T sin θ. It is directed along CO. If θ is small, sinθ = θ and the resultant force due to tension is 2Tθ.

resultant force = 2Tθ

$$= 2T.\frac{dx}{2R} \qquad \left(\because 2\theta = \frac{dx}{R}\right)$$

$$= T.\ \frac{dx}{R} \qquad \ldots (2)$$

For the arc EF to be in equilibrium,

$$\frac{m.dx\,v^2}{R} = \frac{T.dx}{R}$$

$$v^2 = \frac{T}{m}$$

$$\text{or} \quad v = \sqrt{\frac{T}{m}} \qquad \ldots (3)$$

7.2.2 Velocity of longitudinal waves in an elastic medium

Velocity of longitudinal waves in an elastic medium is

$$v = \sqrt{\frac{E}{\rho}} \qquad \ldots(1)$$

where E is the modulus of elasticity, ρ is the density of the medium.

(i) In the case of a solid rod

$$v = \sqrt{\frac{q}{\rho}} \qquad \ldots(2)$$

where q is the Young's modulus of the material of the rod and ρ is the density of the rod.

(ii) In liquids, $v = \sqrt{\frac{k}{\rho}}$ ...(3)

where k is the Bulk modulus and ρ is the density of the liquid.

7.2.3 Newton's formula for the velocity of sound waves in air

Newton assumed that sound waves travel through air under isothermal conditions (i.e) temperature of the medium remains constant.

The change in pressure and volume obeys Boyle's law.

$\therefore$ PV = constant

Differentiating, $P\ .\ dV + V\ .dP = 0$

$$P.\ dV = -V\ dP$$

$$\therefore \; P = \frac{-dP}{\left(\dfrac{dV}{V}\right)} = \frac{\textit{change in pressure}}{\textit{volume strain}}$$

P = k (Volume Elasticity)

Therefore under isothermal condition, $P = k$

$$\upsilon = \sqrt{\frac{k}{\rho}} = \sqrt{\frac{P}{\rho}}$$

where P is the pressure of air and ρ is the density of air. The above equation is known as Newton's formula for the velocity of sound waves in a gas.

At NTP, P = 76 cm of mercury

= $(0.76 \times 13.6 \times 10^3 \times 9.8)\ N\ m^{-2}$

$\rho = 1.293\ \text{kg m}^{-3}$.

∴ Velocity of sound in air at NTP is

$$\upsilon = \sqrt{\frac{0.76 \times 13.6 \times 10^3 \times 9.8}{1.293}} = 280\ \text{m s}^{-1}$$

The experimental value for the velocity of sound in air is 332 m s^{-1}. But the theoretical value of 280 m s^{-1} is 15% less than the experimental value. This discrepancy could not be explained by Newton's formula.

7.2.4 Laplace's correction

The above discrepancy between the observed and calculated values was explained by Laplace in 1816. Sound travels in air as a longitudinal wave. The wave motion is therefore, accompanied by compressions and rarefactions. At compressions the temperature of air rises and at rarefactions, due to expansion, the temperature decreases.

Air is a very poor conductor of heat. Hence at a compression, air cannot lose heat due to radiation and conduction. At a rarefaction it cannot gain heat, during the small interval of time. As a result, the temperature throughout the medium does not remain constant.

Laplace suggested that sound waves travel in air under adiabatic condition and not under isothermal condition.

For an adiabatic change, the relation between pressure and volume is given by

$$P\,V^{\gamma} = \text{constant}$$

where $\gamma = \left(\frac{C_P}{C_V}\right)$ is the ratio of two specific heat capacities of the gas.

Differentiating

$$P\gamma\, V^{\gamma-1} \,.\, dV + V^{\gamma}\, dP = 0$$

$$P\gamma = \frac{-V^{\gamma}\,dP}{V^{\gamma-1}\,dV}$$

$$P\,\gamma = -V.\frac{dP}{dV}$$

$$P\,\gamma = \frac{-dP}{\left(\frac{dV}{V}\right)} = k$$

$\therefore$ $P\,\gamma = k$ (Volume elasticity)

Therefore under adiabatic condition

velocity of sound $\upsilon = \sqrt{\frac{k}{\rho}} = \sqrt{\frac{\gamma P}{\rho}}$

This is Laplace's corrected formula.

For air at NTP

$\gamma = 1.41$, $\rho = 1.293$ kg m^{-3}

$$\therefore \quad \upsilon = \sqrt{\gamma}\,\sqrt{\frac{P}{\rho}} = \sqrt{1.41} \times 280 = 331.3 \text{ ms}^{-1}$$

This result agrees with the experimental value of 332 ms^{-1}.

7.2.5 Factors affecting velocity of sound in gases

(i) Effect of pressure

If the temperature of the gas remains constant, then by Boyle's law PV = constant

i.e $P \,.\, \frac{m}{\rho} = \text{constant}$

$\frac{P}{\rho}$ is a constant, when mass (m) of a gas is constant. If the pressure changes from P to P' then the corresponding density also will change from ρ to ρ' such that $\frac{P}{\rho}$ is a constant.

In Laplace's formula $\sqrt{\frac{\gamma P}{\rho}}$ is also a constant. Therefore the *velocity of sound in a gas is independent of the change in pressure provided the temperature remains constant.*

(ii) Effect of temperature

For a gas, $PV = RT$

$$P.\frac{m}{\rho} = RT$$

$$\text{or } \frac{P}{\rho} = \frac{RT}{m}$$

where m is the mass of the gas, T is the absolute temperature and R is the gas constant.

Therefore $v = \sqrt{\frac{\gamma RT}{m}}$

It is clear that the velocity of sound in a gas is directly proportional to the square root of its absolute temperature.

Let v_o and v_t be the velocity of sound at $0^o C$ and $t^o C$ respectively. Then, from the above equation,

$$v_o = \sqrt{\frac{\gamma R}{m}} \times \sqrt{273}$$

$$v_t = \sqrt{\frac{\gamma R}{m}} \times \sqrt{273+t}$$

$$\therefore \frac{v_t}{v_o} = \sqrt{\frac{273+t}{273}}$$

$$\therefore v_t = v_o \left(1+\frac{t}{273}\right)^{1/2}$$

Using binomial expansion and neglecting higher powers we get,

$$v_t = v_o \left(1+\frac{1}{2}.\frac{t}{273}\right)$$

$$v_t = v_o \left(1+\frac{t}{546}\right)$$

Since $v_o = 331\, m\, s^{-1}$ at $0^o C$

$$v_t = 331 + 0.61\ m\, s^{-1}$$

Thus the velocity of sound in air increases by 0.61 m s^{-1} per degree centigrade rise in temperature.

(iii) Effect of density

Consider two different gases at the same temperature and pressure with different densities. The velocity of sound in two gases are given by

$$v_1 = \sqrt{\frac{\gamma_1 P}{\rho_1}} \quad \text{and} \quad v_2 = \sqrt{\frac{\gamma_2 P}{\rho_2}}$$

$$\therefore \frac{v_1}{v_2} = \sqrt{\frac{\gamma_1}{\gamma_2} \cdot \frac{\rho_2}{\rho_1}}$$

For gases having same value of γ, $\frac{v_1}{v_2} = \sqrt{\frac{\rho_2}{\rho_1}}$

The velocity of sound in a gas is inversely proportional to the square root of the density of the gas.

(iv) Effect of humidity

When the humidity of air increases, the amount of water vapour present in it also increases and hence its density decreases, because the density of water vapour is less than that of dry air. Since velocity of sound is inversely proportional to the square root of density, the sound travels faster in moist air than in dry air. Due to this reason it can be observed that on a rainy day sound travels faster.

(v) Effect of wind

The velocity of sound in air is affected by wind. If the wind blows with the velocity w along the direction of sound, then the velocity of sound increases to $v + w$. If the wind blows in the opposite direction to the direction of sound, then the velocity of sound decreases to $v - w$. If the wind blows at an angle θ with the direction of sound, the effective velocity of sound will be $(v + w \cos \theta)$.

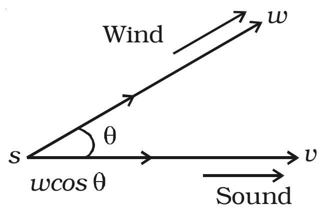


Fig. 7.5 Effect of wind

Note: In a medium, sound waves of different frequencies or wavelengths travel with the same velocity. Hence there is no effect of frequency on the velocity of sound.

Table 7.1 Velocity of sound in various media
(NOT FOR EXAMINATION)

	Medium	Velocity (ms^{-1})
Gases	Air 0° C	331
	Air 20° C	343
	Helium	965
	Hydrogen	1284
Liquids	Water 0° C	1402
	Water at 20° C	1482
	Sea water	1522
Solids	Aluminum	6420
	Steel	5921
	Granite	6000

7.3 Progressive wave

A progressive wave is defined as the onward transmission of the vibratory motion of a body in an elastic medium from one particle to the successive particle.

7.3.1 Equation of a plane progressive wave

An equation can be formed to represent generally the displacement of a vibrating particle in a medium through which a wave passes. Thus each particle of a progressive wave executes simple harmonic motion of the same period and amplitude differing in phase from each other.

Let us assume that a progressive wave travels from the origin O along the positive direction of X axis, from left to right (Fig. 7.6). The displacement of a particle at a given instant is

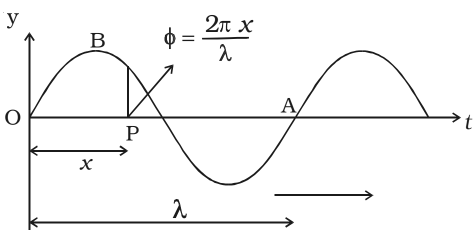


Fig. 7.6 Plane Progressive wave

$$y = a \sin \omega t \qquad \dots (1)$$

where a is the amplitude of the vibration of the particle and $\omega = 2\pi n$.

The displacement of the particle P at a distance x from O at a given instant is given by,

$$y = a \sin(\omega t - \phi) \quad \text{... (2)}$$

If the two particles are separated by a distance λ, they will differ by a phase of 2π. Therefore, the phase ϕ of the particle P at a distance x is $\phi = \frac{2\pi}{\lambda}.x$

$$y = a \sin\left(\omega t - \frac{2\pi x}{\lambda}\right) \quad \text{...(3)}$$

Since $\omega = 2\pi n = 2\pi\frac{v}{\lambda}$, the equation is given by

$$y = a \sin\left(\frac{2\pi vt}{\lambda} - \frac{2\pi x}{\lambda}\right)$$

$$y = a \sin\frac{2\pi}{\lambda}(vt - x) \quad \text{...(4)}$$

Since $\omega = \frac{2\pi}{T}$, the eqn. (3) can also be written as

$$y = a \sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right) \quad \text{...(5)}$$

If the wave travels in opposite direction, the equation becomes.

$$y = a \sin 2\pi\left(\frac{t}{T} + \frac{x}{\lambda}\right) \quad \text{...(6)}$$

(i) Variation of phase with time

The phase changes continuously with time at a constant distance.

At a given distance x from O let ϕ_1 and ϕ_2 be the phase of a particle at time t_1 and t_2 respectively.

$$\phi_1 = 2\pi\left(\frac{t_1}{T} - \frac{x}{\lambda}\right)$$

$$\phi_2 = 2\pi\left(\frac{t_2}{T} - \frac{x}{\lambda}\right)$$

$$\therefore \phi_2 - \phi_1 = 2\pi\left(\frac{t_2}{T} - \frac{t_1}{T}\right) = \frac{2\pi}{T}(t_2 - t_1)$$

$$\Delta\phi = \frac{2\pi}{T}\Delta t$$

This is the phase change $\Delta\phi$ of a particle in time interval Δt. If $\Delta t = T$, $\Delta\phi = 2\pi$. This shows that after a time period T, the phase of a particle becomes the same.

(ii) Variation of phase with distance

At a given time t phase changes periodically with distance x. Let ϕ_1 and ϕ_2 be the phase of two particles at distance x_1 and x_2 respectively from the origin at a time t.

Then $$\phi_1 = 2\pi\left(\frac{t}{T} - \frac{x_1}{\lambda}\right)$$

$$\phi_2 = 2\pi\left(\frac{t}{T} - \frac{x_2}{\lambda}\right)$$

$$\therefore \phi_2 - \phi_1 = -\frac{2\pi}{\lambda}(x_2 - x_1)$$

$$\therefore \Delta\phi = -\frac{2\pi}{\lambda}\Delta x$$

The negative sign indicates that the forward points lag in phase when the wave travels from left to right.

When $\Delta x = \lambda$, $\Delta\phi = 2\pi$, the phase difference between two particles having a path difference λ is 2π.

7.3.2 Characteristics of progressive wave

1. Each particle of the medium executes vibration about its mean position. The disturbance progresses onward from one particle to another.

2. The particles of the medium vibrate with same amplitude about their mean positions.

3. Each successive particle of the medium performs a motion similar to that of its predecessor along the propagation of the wave, but later in time.

4. The phase of every particle changes from 0 to 2π.

5. No particle remains permanently at rest. Twice during each

vibration, the particles are momentarily at rest at extreme positions, different particles attain the position at different time.

6. Transverse progressive waves are characterised by crests and troughs. Longitudinal waves are characterised by compressions and rarefactions.

7. There is a transfer of energy across the medium in the direction of propagation of progressive wave.

8. All the particles have the same maximum velocity when they pass through the mean position.

9. The displacement, velocity and acceleration of the particle separated by $m\lambda$ are the same, where m is an integer.

7.3.3 Intensity and sound level

If we hear the sound produced by violin, flute or harmonium, we get a pleasing sensation in the ear, whereas the sound produced by a gun, horn of a motor car etc. produce unpleasant sensation in the ear.

The loudness of a sound depends on intensity of sound wave and sensitivity of the ear.

The intensity is defined as the amount of energy crossing per unit area per unit time perpendicular to the direction of propagation of the wave.

Intensity is measured in W m^{-2}.

The intensity of sound depends on (i) Amplitude of the source ($I \propto a^2$), (ii) Surface area of the source ($I \propto A$), (iii) Density of the medium ($I \propto \rho$), (iv) Frequency of the source ($I \propto n^2$) and (v) Distance of the observer from the source ($I \propto \frac{1}{r^2}$).

The lowest intensity of sound that can be perceived by the human ear is called *threshold of hearing*. It is denoted by I_o.

For sound of frequency 1 KHz, $I_o = 10^{-12}$ W m^{-2}. The level of sound intensity is measured in decibel. According to Weber-Fechner law,

$$\text{decibel level } (\beta) = 10 \log_{10} \left[\frac{I}{I_o}\right]$$

where I_o is taken as 10^{-12} W m^{-2} which corresponds to the lowest sound intensity that can be heard. Its level is 0 dB. I is the maximum intensity that an ear can tolerate which is 1W m^{-2} equal to 120 dB.

$$\beta = 10 \log_{10} \left(\frac{1}{10^{-12}}\right)$$

$$\beta = 10 \log_{10} (10^{12})$$

$$\beta = 120 \text{ dB.}$$

Table 7.2 gives the decibel value and power density (intensity) for various sources.

Table 7.2 Intensity of sound sources
(NOT FOR EXAMINATION)

Source of sound	Sound intensity(dB)	Intensity ($W m^{-2}$)
Threshold of pain	120	1
Busy traffic	70	10^{-5}
Conversation	65	3.2×10^{-6}
Quiet car	50	10^{-7}
Quiet Radio	40	10^{-8}
Whisper	20	10^{-10}
Rustle of leaves	10	10^{-11}
Threshold of hearing	0	10^{-12}

7.4. Reflection of sound

Take two metal tubes A and B. Keep one end of each tube on a metal plate as shown in Fig. 7.7. Place a wrist watch at the open end of the tube A and interpose a cardboard between A and B. Now at a particular inclination of the tube B with the cardboard, ticking of the watch is clearly heard. The angle of reflection made by the tube B with the cardboard is equal to the angle of incidence made by the tube A with the cardboard.

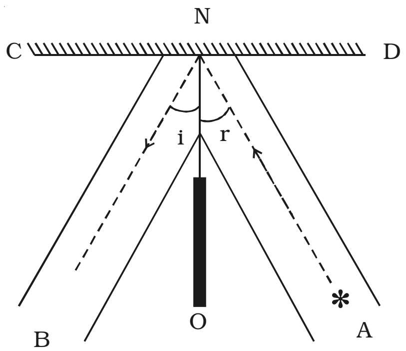


Fig. 7.7 Reflection of sound

7.4.1 Applications of reflection of sound waves

(i) Whispering gallery **:** The famous whispering gallery at

St. Paul's Cathedral is a circular shaped chamber whose walls repeatedly reflect sound waves round the gallery, so that a person talking quietly at one end can be heard distinctly at the other end. This is due to multiple reflections of sound waves from the curved walls (Fig. 7.8).

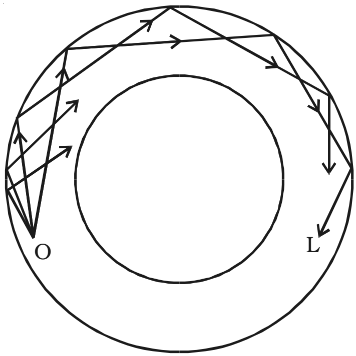


Fig. 7.8 Multiple reflections in the whispering gallery

(ii) Stethoscope : Stethoscope is an instrument used by physicians to listen to the sounds produced by various parts of the body. It consists of a long tube made of rubber or metal. When sound pulses pass through one end of the tube, the pulses get concentrated to the other end due to several reflections on the inner surface of the tube. Using this doctors hear the patients' heart beat as concentrated rays.

(iii) Echo : Echoes are sound waves reflected from a reflecting surface at a distance from the listener. Due to persistence of hearing, we keep hearing the sound for $\frac{1}{10}$th of a second, even after the sounding source has stopped vibrating. Assuming the velocity of sound as 340 ms^{-1}, if the sound reaches the obstacle and returns after 0.1 second, the total distance covered is 34 m. No echo is heard if the reflecting obstacle is less than 17 m away from the source.

7.5 Refraction of sound

This is explained with a rubber bag filled with carbon-di-oxide as shown in Fig. 7.9. The velocity of sound in carbon-di-oxide is less than that in air and hence the bag acts as a lens. If a whistle is used as a source S, the sound passes through the lens and converges at O which is located with the help of flame. The flame will be disturbed only at the point O.

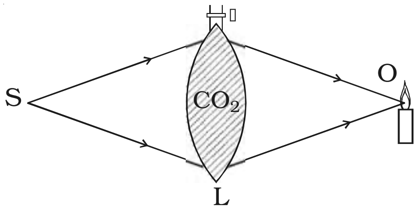


Fig. 7.9 Refraction of sound

When sound travels from one medium to another, it undergoes refraction.

7.5.1 Applications of refraction of sound

It is easier to hear the sound during night than during day-time.

During day time, the upper layers of air are cooler than the layers of air near the surface of the Earth. During night, the layers of air near the Earth are cooler than the upper layers of air. As sound travels faster in hot air, during day-time, the sound waves will be refracted upwards and travel a short distance on the surface of the Earth. On the other hand, during night the sound waves are refracted downwards to the Earth and will travel a long distance.

7.6 Superposition principle

When two waves travel in a medium simultaneously in such a way that each wave represents its separate motion, then the resultant displacement at any point at any time is equal to the vector sum of the individual displacements of the waves.

This principle is illustrated by means of a slinky in the Fig. 7.10(a).

1. In the figure, (i) shows that the two pulses pass each other,
2. In the figure, (ii) shows that they are at some distance apart
3. In the figure, (iii) shows that they overlap partly
4. In the figure, (iv) shows that resultant is maximum

Fig. 7.10 b illustrates the same events but with pulses that are equal and opposite.

If $\vec{Y_1}$ and $\vec{Y_2}$ are the displacements at a point, then the resultant displacement is given by $\vec{Y} = \vec{Y_1} + \vec{Y_2}$.

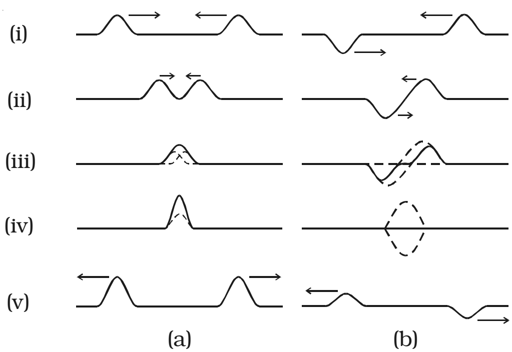


Fig.7.10 Superposition of waves

If $|\vec{Y_1}| = |\vec{Y_2}| = a$, and if the two waves have their displacements in the same direction, then $|\vec{Y}|$ $= a + a = 2a$

If the two waves have their displacements in the opposite direction, then $|\vec{Y}| = a + (-a) = 0$

The principle of superposition of waves is applied in wave phenomena such as interference, beats and stationary waves.

7.6.1 Interference of waves

When two waves of same frequency travelling in the same direction in a medium superpose with each other, their resultant intensity is maximum at some points and minimum at some other points. This phenomenon of superposition is called interference.

Let us consider two simple harmonic waves of same frequency travelling in the same direction. If a_1 and a_2 are the amplitudes of the waves and ϕ is the phase difference between them, then their instantaneous displacements are

$$y_1 = a_1 \sin \omega t \qquad ...(1)$$

$$y_2 = a_2 \sin (\omega t + \phi) \qquad ...(2)$$

According to the principle of superposition, the resultant displacement is represented by

$$y = y_1 + y_2$$

$$= a_1 \sin \omega t + a_2 \sin (\omega t + \phi)$$

$$= a_1 \sin \omega t + a_2 (\sin \omega t . \cos \phi + \cos \omega t . \sin \phi)$$

$$= (a_1 + a_2 \cos \phi) \sin \omega t + a_2 \sin \phi \cos \omega t \qquad ...(3)$$

Put $a_1 + a_2 \cos \phi = A \cos \theta$ ...(4)

$a_2 \sin \phi = A \sin \theta$ ...(5)

where A and θ are constants, then

$$y = A \sin \omega t . \cos \theta + A \cos \omega t . \sin \theta$$

or $y = A \sin (\omega t + \theta)$ ...(6)

This equation gives the resultant displacement with amplitude A. From eqn. (4) and (5)

$$A^2 \cos^2 \theta + A^2 \sin^2 \theta$$

$$= (a_1 + a_2 \cos \phi)^2 + (a_2 \sin \phi)^2$$

$$\therefore A^2 = a_1^2 + a_2^2 + 2a_1 a_2 \cos \phi$$

$$\therefore A = \sqrt{a_1^2 + a_2^2 + 2a_1 a_2 \cos \phi} \qquad ... (7)$$

Also $\tan \theta = \dfrac{a_2 \sin \phi}{a_1 + a_2 \cos \phi}$ ...(8)

We know that intensity is directly proportional to the square of the amplitude

(i.e) $I \alpha A^2$

$$\therefore \quad I \alpha (a_1^2 + a_2^2 + 2a_1a_2 \cos \phi) \qquad \text{... (9)}$$

Special cases

The resultant amplitude A is maximum, when $\cos \phi = 1$ or $\phi = 2m\pi$ where m is an integer (i.e) $I_{max} \alpha (a_1 + a_2)^2$

The resultant amplitude A is minimum when

$\cos \phi = -1$ or $\phi = (2m + 1)\pi$

$I_{min} \alpha (a_1 - a_2)^2$

The points at which interfering waves meet in the same phase $\phi = 2m\pi$ i.e $0, 2\pi, 4\pi, ...$ are points of maximum intensity, where constructive interference takes place. The points at which two interfering waves meet out of phase $\phi = (2m + 1)\pi$ i.e $\pi, 3\pi, ...$ are called points of minimum intensity, where destructive interference takes place.

7.6.2 Experimental demonstration of interference of sound

The phenomenon of interference between two longitudinal waves in air can be demonstrated by Quincke's tube shown in Fig. 7.11.

Quincke's tube consists of U shaped glass tubes *A* and *B*. The tube *SAR* has two openings at *S* and R. The other tube *B* can slide over the tube *A*. *A* sound wave from *S* travels along both the paths *SAR* and *SBR* in opposite directions and meet at *R*.

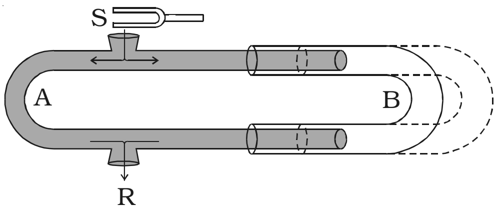


Fig. 7.11 Quincke's Tube

If the path difference between the two waves (i.e) $SAR \sim SBR$ is an integral multiple of wavelength, intensity of sound will be maximum due to constructive interference.

i.e $\quad SAR \sim SBR = m\lambda$

The corresponding phase difference ϕ between the two waves is even multiples of π. (i.e) $\phi = m\, 2\pi$ where $m = 0, 1, 2, 3 ....$

If the tube B is gradually slided over A, a stage is reached when the intensity of sound is zero at R due to destructive interference. Then no sound will be heard at *R*.

If the path difference between the waves is odd multiples of $\frac{\lambda}{2}$, intensity of sound will be minimum.

i.e $SAR \sim SBR = (2m + 1)\frac{\lambda}{2}$

The corresponding phase difference ϕ between the two waves is odd multiples of π. (i.e) $\phi = (2m + 1)\pi$ where m = 0, 1, 2, 3 .....

7.6.3 Beats

When two waves of nearly equal frequencies travelling in a medium along the same direction superimpose upon each other, beats are produced. The amptitude of the resultant sound at a point rises and falls regularly.

The intensity of the resultant sound at a point rises and falls regularly with time. When the intensity rises to maximum we call it as waxing of sound, when it falls to minimum we call it as waning of sound.

The phenomenon of waxing and waning of sound due to interference of two sound waves of nearly equal frequencies are called beats. The number of beats produced per second is called beat frequency, which is equal to the difference in frequencies of two waves.

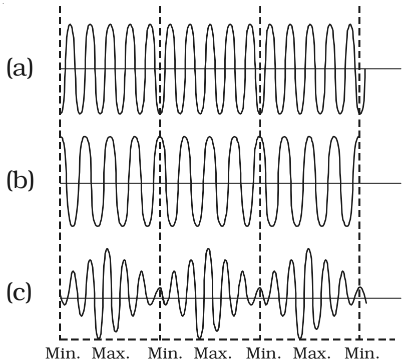


Fig. 7.12 Graphical representation of beats

Analytical method

Let us consider two waves of slightly different frequencies n_1 and n_2 ($n_1 \sim n_2 < 10$) having equal amplitude travelling in a medium in the same direction.

At time t = 0, both waves travel in same phase.

The equations of the two waves are

$y_1 = a \sin \omega_1 t$

$$y_1 = a \sin (2\pi\ n_1)t \qquad ...(1)$$

$$y_2 = a \sin \omega_2\ t$$

$$= a \sin (2\pi\ n_2)t \qquad ...(2)$$

When the two waves superimpose, the resultant displacement is given by

$$y = y_1 + y_2$$

$$y = a \sin (2\pi\ n_1)\ t + a \sin (2\pi\ n_2)\ t \qquad ...(3)$$

Therefore

$$y = 2a \sin 2\pi \left(\frac{n_1+n_2}{2}\right) t \cos 2\pi \left(\frac{n_1-n_2}{2}\right) t \qquad ...(4)$$

Substitute $A = 2\ a \cos 2\pi \left(\frac{n_1-n_2}{2}\right) t$ and $n = \frac{n_1+n_2}{2}$ in equation (4)

$$\therefore \quad y = A \sin 2\pi nt$$

This represents a simple harmonic wave of frequency $n = \frac{n_1+n_2}{2}$ and amplitude A which changes with time.

(i) The resultant amplitude is maximum (i.e) $\pm 2a$, if

$$\cos 2\pi \left[\frac{n_1-n_2}{2}\right] t = \pm 1$$

$$\therefore\ 2\pi \left[\frac{n_1-n_2}{2}\right] t = m\pi$$

(where m = 0, 1, 2 ...) or $(n_1 - n_2)\ t = m$

The first maximum is obtained at $t_1 = 0$

The second maximum is obtained at

$$t_2 = \frac{1}{n_1-n_2}$$

The third maximum at $t_3 = \frac{2}{n_1-n_2}$ and so on.

The time interval between two successive maxima is

$$t_2 - t_1 = t_3 - t_2 = \frac{1}{n_1-n_2}$$

Hence the number of beats produced per second is equal to the reciprocal of the time interval between two successive maxima.

(ii) The resultant amplitude is minimum (i.e) equal to zero, if

$$\cos 2\pi \left(\frac{n_1 - n_2}{2}\right) t = 0$$

(i.e) $2\pi \left(\frac{n_1 - n_2}{2}\right) t = \frac{\pi}{2} + m\pi = (2m + 1)\frac{\pi}{2}$ or $(n_1 - n_2)t = \frac{(2m+1)}{2}$

where m = 0, 1, 2 ...

The first minimum is obtained at

$$t_1' = \frac{1}{2(n_1 - n_2)}$$

The second minimum is obtained at

$$t_2' = \frac{3}{2(n_1 - n_2)}$$

The third minimum is obtained at

$$t_3' = \frac{5}{2(n_1 - n_2)} \text{ and so on}$$

Time interval between two successive minima is

$$t_2' - t_1' = t_3' - t_2' = \frac{1}{n_1 - n_2}$$

Hence, the number of beats produced per second is equal to the reciprocal of time interval between two successive minima.

7.6.4 Uses of beats

(i) The phenomenon of beats is useful in tuning two vibrating bodies in unison. For example, a sonometer wire can be tuned in unison with a tuning fork by observing the beats. When an excited tuning fork is kept on the sonometer and if the sonometer wire is also excited, beats are heard, when the frequencies are nearly equal. If the length of the wire is adjusted carefully so that the number of beats gradually decreases to zero, then the two are said to be in unison. Most of the musical instruments are made to be in unison based on this method.

(ii) The frequency of a tuning fork can be found using beats. A standard tuning fork of frequency N is excited along with the experimental fork. If the number of beats per second is n, then the frequency of experimental tuning fork is $N \pm n$. The experimental tuning fork is then loaded with a little bees' wax, thereby decreasing its frequency. Now the observations are repeated. If the number of beats increases, then the frequency of the experimental tuning fork is N-n, and if the number of beats decreases its frequency is N + n.

7.6.5 Stationary waves

When two progressive waves of same amplitude and wavelength travelling along a straight line in opposite directions superimpose on each other, stationary waves are formed.

Analytical method

Let us consider a progressive wave of amplitude a and wavelength λ travelling in the direction of X axis.

$$y_1 = a \sin 2\pi \left(\frac{t}{T} - \frac{x}{\lambda}\right) \quad .....(1)$$

This wave is reflected from a free end and it travels in the negative direction of X axis, then

$$y_2 = a \sin 2\pi \left(\frac{t}{T} + \frac{x}{\lambda}\right) \quad .....(2)$$

According to principle of superposition, the resultant displacement is

$$y = y_1 + y_2$$

$$= a \left[\sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right) + \sin 2\pi\left(\frac{t}{T} + \frac{x}{\lambda}\right)\right]$$

$$= a \left[2 \sin \frac{2\pi t}{T} \cos \frac{2\pi x}{\lambda}\right]$$

$$\therefore y = 2a \cos \frac{2\pi x}{\lambda} \sin \frac{2\pi t}{T} \quad ...(3)$$

This is the equation of a stationary wave.

(i) At points where $x = 0, \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2}$, the values of $\cos \frac{2\pi x}{\lambda} = \pm 1$

$\therefore A = \pm 2a$. At these points the resultant amplitude is maximum. They are called *antinodes* (Fig. 7.13).

(ii) At points where $x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}$ .... the values of $\cos \frac{2\pi x}{\lambda} = 0$.

$\therefore A = 0$. The resultant amplitude is zero at these points. They are called *nodes* (Fig. 7.16).

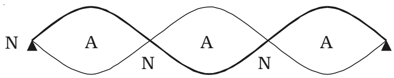


Fig. 7.13 Stationary waves

The distance between any two successive antinodes or nodes is equal to $\frac{\lambda}{2}$ and the distance between an antinode and a node is $\frac{\lambda}{4}$.

(iii) When t = 0, $\frac{T}{2}$, T, $\frac{3T}{2}$, 2T, ... then $\sin\frac{2\pi t}{T} = 0$, the displacement is zero.

(iv) When t = $\frac{T}{4}$, $\frac{3T}{4}$, $\frac{5T}{4}$ etc, ... $\sin\frac{2\pi t}{T} = \pm 1$, the displacement is maximum.

7.6.6 Characteristics of stationary waves

1. The waveform remains stationary.

2. Nodes and antinodes are formed alternately.

3. The points where displacement is zero are called nodes and the points where the displacement is maximum are called antinodes.

4. Pressure changes are maximum at nodes and minimum at antinodes.

5. All the particles except those at the nodes, execute simple harmonic motions of same period.

6. Amplitude of each particle is not the same, it is maximum at antinodes decreases gradually and is zero at the nodes.

7. The velocity of the particles at the nodes is zero. It increases gradually and is maximum at the antinodes.

8. Distance between any two consecutive nodes or antinodes is equal to $\frac{\lambda}{2}$, whereas the distance between a node and its adjacent antinode is equal to $\frac{\lambda}{4}$.

9. There is no transfer of energy. All the particles of the medium pass through their mean position simultaneously twice during each vibration.

10. Particles in the same segment vibrate in the same phase and

between the neighbouring segments, the particles vibrate in opposite phase.

7.7 Standing waves in strings

In musical instruments like sitar, violin, etc. sound is produced due to the vibrations of the stretched strings. Here, we shall discuss the different modes of vibrations of a string which is rigidly fixed at both ends.

When a string under tension is set into vibration, a transverse progressive wave moves towards the end of the wire and gets reflected. Thus stationary waves are formed.

7.7.1 Sonometer

The sonometer consists of a hollow sounding box about a metre long. One end of a thin metallic wire of uniform cross-section is fixed to a hook and the other end is passed over a pulley and attached to a weight hanger as shown in Fig. 7.14. The wire is stretched over two knife edges P and Q by adding sufficient weights on the hanger. The distance between the two knife edges can be adjusted to change the vibrating length of the wire.

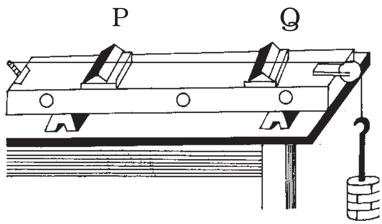


Fig. 7.14 Sonometer

A transverse stationary wave is set up in the wire. Since the ends are fixed, nodes are formed at P and Q and antinode is formed in the middle.

The length of the vibrating segment is $l = \lambda/2$

$\therefore \lambda = 2l$. If n is the frequency of vibrating segment, then

$$n = \frac{v}{\lambda} = \frac{v}{2l} \quad \ldots(1)$$

We know that $v = \sqrt{\frac{T}{m}}$ where T is the tension and m is the mass per unit length of the wire.

$$\therefore n = \frac{1}{2l}\sqrt{\frac{T}{m}} \quad \ldots(2)$$

Modes of vibration of stretched string

(i) Fundamental frequency

If a wire is stretched between two points, a transverse wave travels along the wire and is reflected at the fixed end. A transverse stationary wave is thus formed as shown in Fig. 7.15.

When a wire AB of length l is made to vibrate in one segment then

$$l = \frac{\lambda_1}{2}.$$

$\therefore \lambda_1 = 2l$. This gives the lowest frequency called fundamental frequency $n_1 = \frac{v}{\lambda_1}$

$$\therefore\ n_1 = \frac{1}{2l}\sqrt{\frac{T}{m}} \qquad \ldots(3)$$

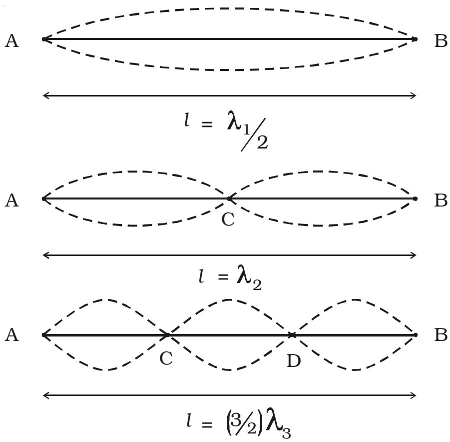


Fig. 7.15 Fundamental and overtones in stretched string

(ii) Overtones in stretched string

If the wire AB is made to vibrate in two segments then $l = \frac{\lambda_2}{2} + \frac{\lambda_2}{2}$

$$\therefore\ \lambda_2 = l.$$

But, $n_2 = \frac{v}{\lambda_2}$ $\qquad \therefore\ n_2 = \frac{1}{l}\sqrt{\frac{T}{m}} = 2n_1 \qquad \ldots(4)$

n_2 is the frequency of the first overtone.

Since the frequency is equal to twice the fundamental, it is also known as second harmonic.

Similarly, higher overtones are produced, if the wire vibrates with more segments. If there are P segments, the length of each segment is

$$\frac{l}{P} = \frac{\lambda_P}{2} \quad \text{or} \quad \lambda_p = \frac{2l}{P}$$

$$\therefore\ \text{Frequency } n_P = \frac{P}{2l}\sqrt{\frac{T}{m}} = P\,n_1 \qquad \ldots(5)$$

(i.e) P^{th} harmonic corresponds to $(P-1)^{th}$ overtone.

7.7.2 Laws of transverse vibrations of stretched strings

The laws of transverse vibrations of stretched strings are (i) the law of length (ii) law of tension and (iii) the law of mass.

(i) For a given wire (m is constant), when T is constant, the fundamental frequency of vibration is inversely proportional to the vibrating length (i.e)

$$n \,\alpha\, \frac{1}{l} \text{ or } nl = \text{constant.}$$

(ii) For constant l and m, the fundamental frequency is directly proportional to the square root of the tension (i.e) $n \,\alpha\, \sqrt{T}$.

(iii) For constant l and T, the fundamental frequency varies inversely as the square root of the mass per unit length of the wire (i.e) $n \,\alpha\, \frac{1}{\sqrt{m}}$.

7.8 Vibrations of air column in pipes

Musical wind instruments like flute, clarinet etc. are based on the principle of vibrations of air columns. Due to the superposition of the incident wave and the reflected wave, longitudinal stationary waves are formed in the pipe.

7.8.1 Organ pipes

Organ pipes are musical instruments which are used to produce musical sound by blowing air into the pipe. Organ pipes are two types (i) closed organ pipes, closed at one end (ii) open organ pipe, open at both ends.

(i) Closed organ pipe : If the air is blown lightly at the open end of the closed organ pipe, then the air column vibrates (Fig. 7.16a) in the fundamental mode. There is a node at the closed end and an antinode at the open end. If l is the length of the tube,

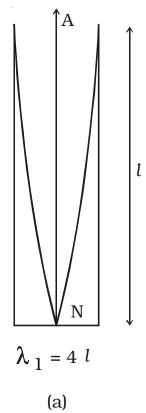


Fig. 7.16a Statinary waves in a closed pipe (Fundamental mode)

$$l = \frac{\lambda_1}{4} \text{ or } \lambda_1 = 4l \quad \dots (1)$$

If n_1 is the fundamental frequency of the vibrations and v is the velocity of sound in air, then

$$n_1 = \frac{v}{\lambda_1} = \frac{v}{4l} \quad \text{... (2)}$$

If air is blown strongly at the open end, frequencies higher than fundamental frequency can be produced. They are called overtones. Fig.7.16b & Fig.7.16c shows the mode of vibration with two or more nodes and antinodes.

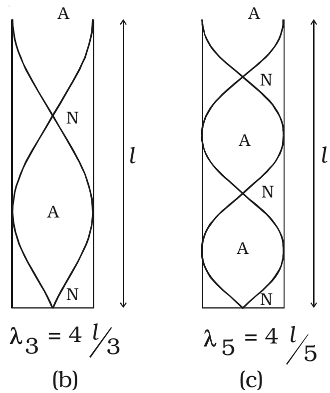


Fig. 7.16b & c Overtones in closed pipe

$$l = \frac{3\lambda_3}{4} \text{ or } \lambda_3 = \frac{4l}{3} \quad \text{...(3)}$$

$$\therefore\ n_3 = \frac{v}{\lambda_3} = \frac{3v}{4l} = 3n_1 \quad \text{...(4)}$$

This is the first overtone or third harmonic.

Similarly $n_5 = \frac{5v}{4l} = 5n_1$ ...(5)

This is called as second overtone or fifth harmonic.

Therefore the frequency of pth overtone is $(2p + 1)\,n_1$ where n_1 is the fundamental frequency. In a closed pipe only odd harmonics are produced. The frequencies of harmonics are in the ratio of 1 : 3 : 5.....

(ii) Open organ pipe - When air is blown into the open organ pipe, the air column vibrates in the fundamental mode Fig. 7.17a. Antinodes are formed at the ends and a node is formed in the middle of the pipe. If l is the length of the pipe, then

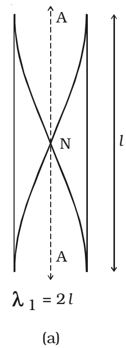


Fig. 7.17a Stationary waves in an open pipe (Fundamental mode)

$$l = \frac{\lambda_1}{2} \text{ or } \lambda_1 = 2l \quad \text{...(1)}$$

$$v = n_1\lambda_1 = n_1 2l$$

The fundamental frequency

$$n_1 = \frac{v}{2l} \quad \text{...(2)}$$

In the next mode of vibration additional nodes and antinodes are formed as shown in

Fig. 7.17b and Fig.7.17c.

$$l = \lambda_2 \text{ or } v = n_2\lambda_2 = n_2 \,.\, l.$$

$$\therefore\ n_2 = \left(\frac{v}{l}\right) = 2n_1 \qquad \text{...(3)}$$

This is the first overtone or second harmonic.

$$\text{Similarly, } n_3 = \frac{v}{\lambda_3} = \frac{3v}{2l} = 3n_1 \qquad \text{...(4)}$$

This is the second overtone or third harmonic.

Therefore the frequency of Pth overtone is $(P + 1)\, n_1$ where n_1 is the fundamental frequency. The frequencies of harmonics are in the ratio of 1 : 2 : 3 ....

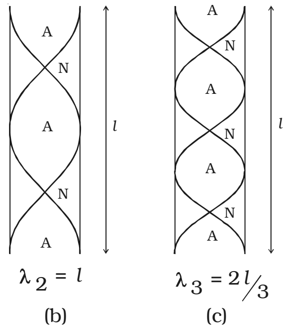


Fig. 7.17b & c Overtones in an open pipe

7.9 Resonance air column apparatus

The resonance air column apparatus consists of a glass tube G about one metre in length (Fig. 7.18) whose lower end is connected to a reservoir R by a rubber tube.

The glass tube is mounted on a vertical stand with a scale attached to it. The glass tube is partly filled with water. The level of water in the tube can be adjusted by raising or lowering the reservoir.

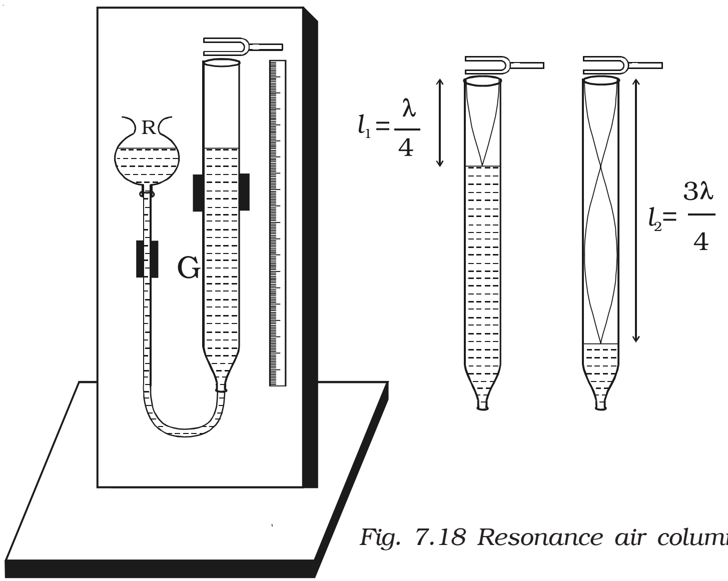


Fig. 7.18 Resonance air column apparatus

A vibrating tuning fork of frequency n is held near the open end of the tube. The length of the air column is adjusted by changing the water level. The air column of the tube acts like a closed organ pipe. When this air column resonates with the frequency of the fork the intensity of sound is maximum.

Here longitudinal stationary wave is formed with node at the water surface and an antinode near the open end. If l_1 is the length of the resonating air column

$$\frac{\lambda}{4} = l_1 + e \qquad ..(1)$$

where e is the end correction.

The length of air column is increased until it resonates again with the tuning fork. If l_2 is the length of the air column.

$$\frac{3\lambda}{4} = l_2 + e \qquad ...(2)$$

From equations (1) and (2)

$$\frac{\lambda}{2} = (l_2 - l_1) \qquad ...(3)$$

The velocity of sound in air at room temperature

$$v = n\lambda = 2n\ (l_2 - l_1) \qquad ...(4)$$

End correction

The antinode is not exactly formed at the open end, but at a small distance above the open end. This is called the end correction.

As $l_1 + e = \frac{\lambda}{4}$ and $l_2 + e = \frac{3\lambda}{4}$

$$e = \frac{(l_2 - 3l_1)}{2}$$

It is found that $e = 0.61r$, where r is the radius of the glass tube.

7.10 Doppler effect

The whistle of a fast moving train appears to increase in pitch as it approaches a stationary observer and it appears to decrease as the train moves away from the observer. This apparent change in frequency was first observed and explained by Doppler in 1845.

The phenomenon of the apparent change in the frequency of sound due to the relative motion between the source of sound and the observer is called Doppler effect.

The apparent frequency due to Doppler effect for different cases can be deduced as follows.

(i) Both source and observer at rest

Suppose S and O are the positions of the source and the observer respectively. Let n be the frequency of the sound and v be the velocity of sound. In one second, n waves produced by the source travel a distance SO = v (Fig. 7.19a).

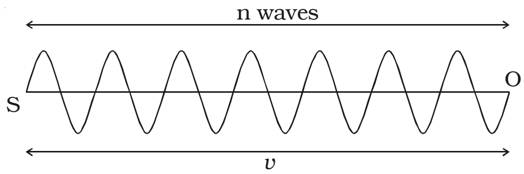


Fig. 7.19a Both source and observer at rest

The wavelength is $\lambda = \frac{v}{n}$.

(ii) When the source moves towards the stationary observer

If the source moves with a velocity v_s towards the stationary observer, then after one second, the source will reach S′, such that SS′ = v_s. Now n waves emitted by the source will occupy a distance of $(v - v_s)$ only as shown in Fig. 7.19b.

Therefore the apparent wavelength of the sound is

$$\lambda' = \frac{v - v_s}{n}$$

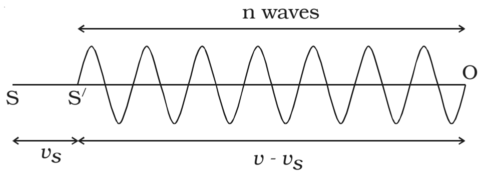


Fig. 7.19b Source moves towards observer at rest

The apparent frequency

$$n' = \frac{v}{\lambda'} = \left(\frac{v}{v - v_s}\right)n \qquad \ldots(1)$$

As $n' > n$, the pitch of the sound appears to increase.

When the source moves away from the stationary observer

If the source moves away from the stationary observer with velocity v_s, the apparent frequency will be given by

$$n' = \left(\frac{v}{v - (-v_s)}\right)n = \left(\frac{v}{v + v_s}\right)n \qquad \ldots(2)$$

As $n' < n$, the pitch of the sound appears to decrease.

(iii) Source is at rest and observer in motion

S and O represent the positions of source and observer respectively. The source S emits n waves per second having a wavelength $\lambda = \frac{v}{n}$. Consider a point A such that OA contains n waves which crosses the ear of the observer in one second (Fig. 7.20a). (i.e) when the first wave is at the point A, the n^{th} wave will be at O, where the observer is situated.

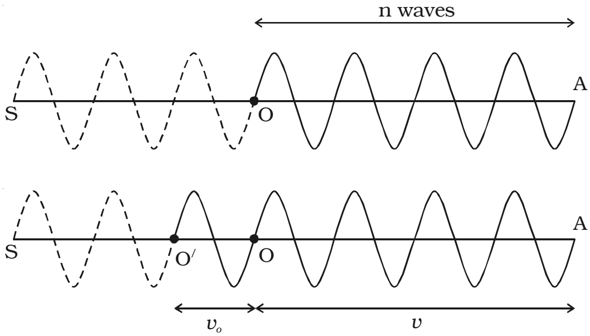


Fig. 7.20a & 7.20b Observer is moving towards a source at rest

When the observer moves towards the stationary source

Suppose the observer is moving towards the stationary source with velocity v_o. After one second the observer will reach the point O′ such that OO′ = v_o. The number of waves crossing the observer will be n waves in the distance OA in addition to the number of waves in the distance OO′ which is equal to $\frac{v_o}{\lambda}$ as shown in Fig. 7.20b.

Therefore, the apparent frequency of sound is

$$n' = n + \frac{v_o}{\lambda} = n + \left(\frac{v_o}{v}\right) n$$

$$\therefore\ n' = \left(\frac{v+v_o}{v}\right)n \qquad \ldots(3)$$

As $n' > n$, the pitch of the sound appears to increase.

When the observer moves away from the stationary source

$$n' = \left[\frac{v+(-v_o)}{v}\right]n$$

$$n' = \left(\frac{v - v_o}{v}\right)n \qquad ...(4)$$

As $n' < n$, the pitch of sound appears to decrease.

Note : If the source and the observer move along the same direction, the equation for apparent frequency is

$$n' = \left(\frac{v - v_o}{v - v_S}\right)n \qquad ...(5)$$

Suppose the wind is moving with a velocity W in the direction of propagation of sound, the apparent frequency is

$$n' = \left(\frac{v + W - v_o}{v + W - v_s}\right)n \qquad ...(6)$$

Applications of Doppler effect

(i) To measure the speed of an automobile

An electromagnetic wave is emitted by a source attached to a police car. The wave is reflected by a moving vechicle, which acts as a moving source. There is a shift in the frequency of the reflected wave. From the frequency shift using beats, the speeding vehicles are trapped by the police.

(ii) Tracking a satellite

The frequency of radio waves emitted by a satellite decreases as the satellite passes away from the Earth. The frequency received by the Earth station, combined with a constant frequency generated in the station gives the beat frequency. Using this, a satellite is tracked.

(iii) RADAR (RADIO DETECTION AND RANGING)

A RADAR sends high frequency radiowaves towards an aeroplane. The reflected waves are detected by the receiver of the radar station. The difference in frequency is used to determine the speed of an aeroplane.

(iv) SONAR (SOUND NAVIGATION AND RANGING)

Sound waves generated from a ship fitted with SONAR are transmitted in water towards an approaching submarine. The frequency of the reflected waves is measured and hence the speed of the submarine is calculated.

Solved Problems

7.1 What is the distance travelled by sound in air when a tuning fork of frequency 256 Hz completes 25 vibrations? The speed of sound in air is 343 m s^{-1}.

Data : $v = 343\ m\ s^{-1}$, $n = 256$ Hz, $d = ?$

Solution : $v = n\lambda$

$$\therefore \lambda = \frac{343}{256} = 1.3398\ m$$

Wavelength is the distance travelled by the wave in one complete vibration of the tuning fork.

$\therefore$ *Distance travelled by sound wave in 25 vibrations = 25 x 1.3398*

Distance travelled by sound wave is = 33.49 m

7.2 Ultrasonic sound of frequency 100 kHz emitted by a bat is incident on a water surface. Calculate the wavelength of reflected sound and transmitted sound? (speed of sound in air $340\ m\ s^{-1}$ and in water $1486\ m\ s^{-1}$)

Data : $n = 100\ kHz = 10^5\ Hz$, $v_a = 340\ m\ s^{-1}$, $v_w = 1486\ m\ s^{-1}$;

$\lambda_a = ?$, $\lambda_w = ?$

Wavelength of reflected sound $\lambda_a = \frac{v_a}{n}$

$$\lambda_a = \frac{340}{10^5} = 3.4 \times 10^{-3}\ m$$

Wavelength of transmitted sound $\lambda_w = \frac{v_w}{n}$

$$\lambda_w = \frac{1486}{10^5} = 1.486 \times 10^{-2}\ m$$

7.3 A string of mass 0.5 kg and length 50 m is stretched under a tension of 400 N. A transverse wave of frequency 10 Hz travels through the wire. (i) Calculate the wave velocity and wavelength. (ii) How long does the disturbance take to reach the other end?

Data : *m = 0.5 kg, length of the wire = 50m; T = 400 N; n=10 Hz*

$v = ?\ ;\ \lambda = ?\ ;\ t = ?$

Solution : *mass per unit length* $m = \frac{\text{mass of the wire}}{\text{length of the wire}}$

$$m = \frac{0.5}{50} = 0.01\ kg\ m^{-1}$$

Velocity in the stretched string $v = \sqrt{\frac{T}{m}}$

$$v = \sqrt{\frac{400}{0.01}} = 200\ m\ s^{-1}$$

$$v = n\lambda$$

$$200 = 10\lambda$$

$$\therefore \quad \lambda = 20\ m$$

The length of the wire = 50 m

$\therefore$ *Time taken for the transverse wave to travel*

a distance 50 m $= \frac{50}{200} = 0.25\ s$

7.4 Determine the velocity and wavelength of sound of frequency 256 Hz travelling in water of Bulk modulus 0.022×10^{11} Pa

Data : $k = 0.022 \times 10^{11}$ Pa, $\rho = 1000\ kg\ m^{-3}$, $n = 256$ Hz

Solution : *Velocity of sound in water* $v = \sqrt{\frac{k}{\rho}}$

$$v = \sqrt{\frac{0.022 \times 10^{11}}{1000}} = 1483\ ms^{-1}$$

$$\therefore \lambda = \frac{v}{n} = \frac{1483}{256} = 5.79\ m$$

7.5 Calculate the speed of longitudinal wave in air at 27° C (The molecular mass of air is 28.8 g mol^{-1}. γ for air is 1.4, R = 8.314 J mol^{-1} K^{-1})

Data : $m = 28.8 \times 10^{-3}\ kg\ mol^{-1}$, $\gamma = 1.4$,

$R = 8.314\ J\ mol^{-1}\ K^{-1}$, $T = 27°C = 300\ K$

Solution : $v = \sqrt{\frac{\gamma RT}{m}} = \sqrt{\frac{1.4 \times 8.314 \times 300}{28.8 \times 10^{-3}}}$

$$v = 348.2\ m\ s^{-1}$$

7.6 For air at NTP, the density is 0.001293 g cm^{-3}. Calculate the velocity of longitudinal wave (i) using Newton's formula (ii) Laplace's correction

Data : $\gamma = 1.4$, $P = 1.013 \times 10^5$ N m^{-2},

$\rho = 0.001293 \times 10^3$ kg m^{-3}.

Solution : *By Newton's formula the velocity of longitudinal wave*

$$v = \sqrt{\frac{P}{\rho}} = \sqrt{\frac{1.013 \times 10^5}{0.001293 \times 10^3}}$$

$$v = 279.9 \text{ m s}^{-1}$$

By Laplace's formula

$$v = \sqrt{\frac{\gamma P}{\rho}} = \sqrt{\frac{1.4 \times 1.013 \times 10^5}{0.001293 \times 10^3}}$$

$$v = 331.18 \text{ m s}^{-1}$$

7.7 The velocity of sound at 27°C is 347 m s^{-1}. Calculate the velocity of sound in air at 627° C.

Data : $v_{27} = 347$ m s^{-1}, $v_{627} = ?$

Solution : $v \propto \sqrt{T}$

$$\frac{v_{27}}{v_{627}} = \sqrt{\frac{273+27}{273+627}} = \sqrt{\frac{300}{900}}$$

$$\frac{v_{27}}{v_{627}} = \sqrt{\frac{1}{3}}$$

$$\therefore v_{627} = v_{27} \times \sqrt{3} = 347 \times \sqrt{3}$$

$$= 347 \times 1.732 = 601 \text{ m s}^{-1}$$

Velocity of sound in air at 627°C is 601 m s^{-1}

7.8 The equation of a progressive wave is y = 0.50 sin (500 t - 0.025x), where y, t and x are in cm, second and metre. Calculate (i) amplitude (ii) angular frequency (iii) period (iv) wavelength and (v) speed of propagation of wave.

Solution : *The general equation of a progressive wave is given by*

$$y = a \sin\left(\omega t - \frac{2\pi}{\lambda}x\right)$$

given $y = 0.50 \sin (500\, t - 0.025x)$

comparing the two equations,

(i) *amplitude* $a = 0.50 \times 10^{-2}\ m$

(ii) *angular frequency* $\omega = 500\ rad\ s^{-1}$

(iii) *time period* $T = \frac{2\pi}{\omega} = \frac{2\pi}{500} = \frac{\pi}{250} s$

(iv) *wavelength* $\lambda = \frac{2\pi}{0.025} m$

$$\lambda = 80\pi = 251.2\ m$$

(v) *wave velocity* $v = n\lambda$

$$= \frac{250}{\pi} \times 80\pi$$

$$v = 2 \times 10^4\ m\ s^{-1}$$

7.9 A source of sound radiates energy uniformly in all directions at a rate of 2 watt. Find the intensity (i) in $W\ m^{-2}$ and (ii) in decibels, at a point 20 m from the source.

Data : *Power = 2 watt, r = 20 m*

Solution : *Intensity of sound* $I = \frac{\text{Power}}{\text{area}}$

$$I = \frac{2}{4\pi(20)^2}$$

(A spherical surface of radius 20 m with source of sound as centre is imagined)

$$I = 4 \times 10^{-4}\ W\ m^{-2}$$

$$\text{Intensity} = 10 \log_{10}\left(\frac{I}{I_o}\right)$$

$$= 10 \log_{10}\left(\frac{4\times10^{-4}}{10^{-12}}\right) \qquad (\because I_o = 10^{-12})$$

$$= 10 \log_{10} (4 \times 10^8)$$

Intensity = 86 dB

7.10 Two tuning forks A and B when sounded together produce 4 beats. If A is in unison with the 0.96 m length of a sonometer wire under a tension, B is in unison with 0.97 m length of the same wire under same tension. Calculate the frequencies of the forks.

Data : $l_1 = 0.96\ m;\ l_2 = 0.97\ m;\ n_1 = ?;\ n_2 = ?$

$l_1 < l_2 \qquad \therefore n_1 > n_2$

Solution : *Let* $n_1 = n$ *and* $n_2 = n - 4$

According to 1st law of transverse vibrations

$$n_1 l_1 = n_2 l_2$$

$$n \times 0.96 = (n-4) \times 0.97$$

$$n(0.97 - 0.96) = 3.88$$

$$\therefore n = \frac{3.88}{0.01} = 388 \text{ Hz}$$

$$\therefore n_2 = 388 - 4 = 384 \text{ Hz}$$

The frequency of the fork A is $n_1 = 388$ *Hz,*

The frequency of the fork B is $n_2 = 384$ *Hz.*

7.11 A string of length 1 m and mass 5×10^{-4} kg fixed at both ends is under a tension of 20 N. If it vibrates in two segments, determine the frequency of vibration of the string.

Data : *The string vibrates with 2 segments.*

$P = 2$ *loops,* $l = 1$ *m,* $m = 5 \times 10^{-4}$ *kg* m^{-1}*,* $T = 20$ *N*

Solution : *Frequency of vibration* $n = \frac{P}{2l}\sqrt{\frac{T}{m}}$

$$\therefore n = \frac{2}{2 \times 1}\sqrt{\frac{20}{5 \times 10^{-4}}}$$

$$n = 200 \text{ Hz}$$

7.12 A stretched string made of aluminium is vibrating at its fundamental frequency of 512 Hz. What is the fundamental frequency of a second string made from the same material which has a diameter and length twice that of the original and which is subjected to three times the force of the original?

Data : $n = 512$ *Hz, In the second case, tension = 3T, length = 2 l, radius = 2r*

Solution : *Let l be the length, T be the tension and r be the radius of the wire, then*

$$n = \frac{1}{2l}\sqrt{\frac{T}{m}}$$

Mass per unit length can be written as the product of cross-sectional area of the wire and density (i.e) $m = \pi r^2 d$

$$512 = \frac{1}{2l}\sqrt{\frac{T}{\pi r^2 d}} \qquad ....(1)$$

In the second case

$$n = \frac{1}{2 \times 2l}\sqrt{\frac{3T}{\pi (2r)^2 d}} \qquad ....(2)$$

Dividing the second equation by first equation

$$\frac{n}{512} = \frac{1}{2}\sqrt{\frac{3}{(2)^2}} \qquad (i.e)\ n = \frac{512}{4}\sqrt{3} = 222\ Hz$$

7.13 The third overtone of a closed pipe is found to be in unison with the first overtone of an open pipe. Determine the ratio of the lengths of the pipes.

Solution : *Let l_1 and l_2 be the lengths of the closed pipe and open pipe respectively. n_1 and n_2 are their fundamental frequencies.*

For closed pipe $n_1 = \frac{v}{4l_1}$

For open pipe $n_2 = \frac{v}{2l_2}$

Third overtone of closed pipe $= (2P + 1)\ n_1 = (2 \times 3 + 1)\ n_1 = 7n_1$

First overtone of open pipe $= (P + 1)\ n_2 = (1 + 1)\ n_2 = 2n_2$

$$\therefore\ 7n_1 = 2n_2$$

$$7 \times \frac{v}{4l_1} = 2 \times \frac{v}{2l_2}$$

$$\therefore\ \frac{l_1}{l_2} = \frac{7}{4}$$

7.14 The shortest length of air in a resonance tube which resonates with a tuning fork of frequency 256 Hz is 32 cm. The corresponding length for the fork of frequency 384 Hz is 20.8 cm. Calculate the end correction and velocity of sound in air .

Data : $n_1 = 256\ Hz,\ l_1 = 32 \times 10^{-2}\ m$

$n_2 = 384\ Hz,\ l_2 = 20.8 \times 10^{-2}\ m$

Solution : *In a closed pipe* $n = \frac{v}{4(l+e)}$

For the first tuning fork, $256 = \frac{v}{4(32+e)\times 10^{-2}}$ *and* *for the second tuning fork,* $384 = \frac{v}{4(20.8+e)\times10^{-2}}$

Dividing the first equation by second equation,

$$\frac{256}{384} = \frac{20.8+e}{32+e}$$

$\therefore$ $e = 1.6$ *cm.*

$v = 256 \times 4\ (32 + 1.6) \times 10^{-2}$

Velocity of sound in air $v = 344\ m\ s^{-1}$

7.15 A railway engine and a car are moving parallel but in opposite direction with velocities 144 km/hr and 72 km/hr respectively. The frequency of engine's whistle is 500 Hz and the velocity of sound is 340 m s^{-1}. Calculate the frequency of sound heard in the car when (i) the car and engine are approaching each other (ii) both are moving away from each other.

Data : *The velocity of source* v_S = 144 *km/hr and*

the velocity of observer v_o = 72 *km/hr*

$v = 340\ m\ s^{-1}$, $n = 500$ *Hz*

Solution : *(i) When the car and engine approaches each other*

$$n' = \left(\frac{v+v_o}{v-v_S}\right)n$$

$$v_S = \frac{144\times10^3}{60\times60} = 40\ m\ s^{-1}$$

$$v_o = \frac{72\times10^3}{60\times60} = 20\ m\ s^{-1}$$

$$\therefore\ n' = \frac{340+20}{340-40} \times 500$$

The frequency of sound heard is = 600 Hz

(ii) When the car and engine are moving away from each other

$$n'' = \left(\frac{v-v_o}{v+v_S}\right)n$$

$$= \frac{340-20}{340+40} \times 500$$

The frequency of sound heard is = 421 Hz

8. Heat and Thermodynamics

In early days, according to caloric theory of heat, heat was regarded as an invisible and weightless fluid called “caloric”. The two bodies at different temperatures placed in contact attain thermal equilibrium by the exchange of caloric. The caloric flows from the hot body to the cold body, till their temperature becomes equal. However, this theory failed to explain the production of heat due to friction in the experiments conducted by Court Rumford. Rubbing our hands against each other produces heat. Joule’s paddle wheel experiment led to the production of heat by friction. These observations led to the dynamic theory of heat, according to which *heat is a form of energy called thermal energy.*

Every body is made up of molecules. Depending on its nature and temperature, the molecules may possess translatory motion, vibratory motion and rotatory motion about its axis. Each type of motion provides some kinetic energy to the molecules. Heat possessed by a body is the total thermal energy of the body, which is the sum of kinetic energies of all the individual molecules of the body.

Temperature of a body is the degree of hotness or coldness of the body. Heat flows from a body at high temperature to a body at low temperature when they are in contact with each other. Modern concept of temperature follows from zeroth law of thermodynamics. *Temperature is the thermal state of the body, that decides the direction of flow of heat.* Temperature is now regarded as one of the fundamental quantities.

8.1 Kinetic theory of gases

The founder of modern kinetic theory of heat by common consent is Daniel Bernoulli. But the credit for having established it on a firm mathematical basis is due to Clausius and Maxwell in whose hands it attained the present form.

8.1.1 Postulates of Kinetic theory of gases

(1) A gas consists of a very large number of molecules. Each one is a perfectly identical elastic sphere.

(2) The molecules of a gas are in a state of continuous and random motion. They move in all directions with all possible velocities.

(3) The size of each molecule is very small as compared to the distance between them. Hence, the volume occupied by the molecule is negligible in comparison to the volume of the gas.

(4) There is no force of attraction or repulsion between the molecules and the walls of the container.

(5) The collisions of the molecules among themselves and with the walls of the container are perfectly elastic. Therefore, momentum and kinetic energy of the molecules are conserved during collisions.

(6) A molecule moves along a straight line between two successive collisions and the average distance travelled between two successive collisions is called the mean free path of the molecules.

(7) The collisions are almost instantaneous (i.e) the time of collision of two molecules is negligible as compared to the time interval between two successive collisions.

Avogadro number

Avogadro number is defined as the number of molecules present in one mole of a substance. It is constant for all the substances. Its value is 6.023×10^{23}.

8.1.2 Pressure exerted by a gas

The molecules of a gas are in a state of random motion. They continuously collide against the walls of the container. During each collision, momentum is transfered to the walls of the container. The pressure exerted by the gas is due to the continuous collision of the molecules against the walls of the container. Due to this continuous collision, the walls experience a continuous force which is equal to the total momentum imparted to the walls per second. The force experienced per unit area of the walls of the container determines the pressure exerted by the gas.

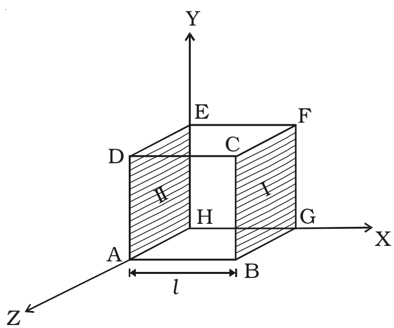


Fig. 8.1 Pressure exerted by a gas

Consider a cubic container of side l containing n molecules of perfect gas moving with velocities C_1, C_2, C_3 ... C_n *(Fig. 8.1)*. A molecule moving with a velocity C_1, will have velocities u_1, v_1 and w_1 as components along the x, y and z axes respectively. Similarly u_2, v_2 and w_2 are the velocity components of the second molecule and so on. Let a molecule P (Fig. 8.2) having velocity C_1 collide against the wall marked I (BCFG) perpendicular to the x-axis. Only the x-component of the velocity of the molecule is relevant for the wall I. Hence momentum of the molecule before collision is mu_1 where m is the mass of the molecule. Since the collision is elastic, the molecule will rebound with the velocity u_1 in the opposite direction. Hence momentum of the molecule after collision is $-mu_1$.

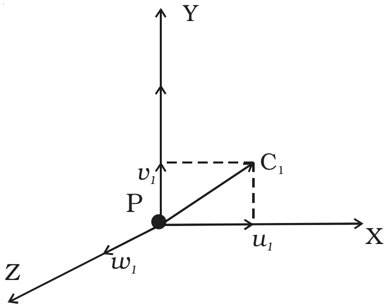


Fig. 8.2 Components of velocity

Change in the momentum of the molecule

= Final momentum - Initial momentum

$= -mu_1 - mu_1 = -2mu_1$

During each successive collision on face I the molecule must travel a distance $2l$ from face I to face II and back to face I.

Time taken between two successive collisions is $= \frac{2l}{u_1}$

$\therefore$ Rate of change of momentum

$$= \frac{\text{Change in the momentum}}{\text{Time taken}}$$

$$= \frac{-2mu_1}{\frac{2l}{u_1}} = \frac{-2mu_1^2}{2\,l} = \frac{-mu_1^2}{l}$$

(i.e) Force exerted on the molecule $= \frac{-mu_1^2}{l}$

$\therefore$ According to Newton's third law of motion, the force exerted by the molecule

$$= -\frac{\left(-mu_1^2\right)}{l} = \frac{mu_1^2}{l}$$

Force exerted by all the n molecules is

$$F_x = \frac{mu_1^2}{l} + \frac{mu_2^2}{l} + \ldots.. + \frac{mu_n^2}{l}$$

Pressure exerted by the molecules

$$P_x = \frac{F_x}{A}$$

$$= \frac{1}{l^2}\left(\frac{mu_1^2}{l} + \frac{mu_2^2}{l} + \ldots.. + \frac{mu_n^2}{l}\right)$$

$$= \frac{m}{l^3}\left(u_1^2 + u_2^2 + \ldots.. + u_n^2\right)$$

Similarly, pressure exerted by the molecules along Y and Z axes are

$$P_y = \frac{m}{l^3}\left(v_1^2 + v_2^2 + \ldots.. + v_n^2\right)$$

$$P_z = \frac{m}{l^3}\left(w_1^2 + w_2^2 + \ldots.. + w_n^2\right)$$

Since the gas exerts the same pressure on all the walls of the container

$$P_x = P_y = P_z = P$$

$$P = \frac{P_x + P_y + p_z}{3}$$

$$P = \frac{1}{3}\frac{m}{l^3}[(u_1^2 + u_2^2 + \ldots + u_n^2) + (v_1^2 + v_2^2 + \ldots. + v_n^2) + (w_1^2 + w_2^2 + \ldots. + w_n^2)]$$

$$P = \frac{1}{3}\frac{m}{l^3}[(u_1^2 + v_1^2 + w_1^2) + (u_2^2 + v_2^2 + w_2^2) + \ldots\ldots + (u_n^2 + v_n^2 + w_n^2)]$$

$$P = \frac{1}{3}\frac{m}{l^3}[C_1^2 + C_2^2 + \ldots. + C_n^2]$$

where $C_1^2 = (u_1^2 + v_1^2 + w_1^2)$

$$P = \frac{1}{3}\frac{mn}{l^3}\left[\frac{C_1^2 + C_2^2 + \ldots. + C_n^2}{n}\right]$$

$$P = \frac{1}{3}\frac{mn}{V} \cdot C^2$$

where C is called the root mean square (RMS) velocity, which is defined as the square root of the mean value of the squares of velocities of individual molecules.

$$\text{(i.e.) } C = \sqrt{\frac{C_1^{\,2} + C_2^{\,2} + ... + C_n^{\,2}}{n}}$$

8.1.3 Relation between the pressure exerted by a gas and the mean kinetic energy of translation per unit volume of the gas

Pressure exerted by unit volume of a gas, $P = \frac{1}{3} mnC^2$

$P = \frac{1}{3} \rho C^2$ ($\because$ mn = mass per unit volume of the gas ; mn = ρ, density of the gas)

Mean kinetic energy of translation per unit volume of the gas

$$E = \frac{1}{2} \rho C^2$$

$$\frac{P}{E} = \frac{\frac{1}{3} \rho C^2}{\frac{1}{2} \rho C^2} = \frac{2}{3}$$

$$P = \frac{2}{3} E$$

8.1.4 Average kinetic energy per molecule of the gas

Let us consider one mole of gas of mass M and volume V.

$$P = \frac{1}{3} \rho C^2$$

$$P = \frac{1}{3} \frac{M}{V} C^2$$

$$PV = \frac{1}{3} MC^2$$

From gas equation

$$PV = RT$$

$$\therefore \quad RT = \frac{1}{3} MC^2$$

$$\frac{3}{2} RT = \frac{1}{2} MC^2$$

(i.e) Average kinetic energy of one mole of the gas is equal to $\frac{3}{2}$RT

Since one mole of the gas contains N number of atoms where N is the Avogadro number we have $M = Nm$

$$\therefore \frac{1}{2} mNC^2 = \frac{3}{2} RT$$

$$\frac{1}{2} mC^2 = \frac{3}{2} \frac{R}{N} T$$

$$= \frac{3}{2} kT \text{ where } k = \frac{R}{N}, \text{ is the Boltzmann constant}$$

Its value is 1.38×10^{-23} J K^{-1}

$\therefore$ *Average kinetic energy per molecule of the gas is equal to* $\frac{3}{2} kT$

Hence, it is clear that the temperature of a gas is the measure of the mean translational kinetic energy per molecule of the gas.

8.2 Degrees of freedom

The number of degrees of freedom of a dynamical system is defined as the total number of co-ordinates or independent variables required to describe the position and configuration of the system.

For translatory motion

(i) A particle moving in a straight line along any one of the axes has one degree of freedom (e.g) Bob of an oscillating simple pendulum.

(ii) A particle moving in a plane (X and Y axes) has two degrees of freedom. (eg) An ant that moves on a floor.

(iii) A particle moving in space (X, Y and Z axes) has three degrees of freedom. (eg) a bird that flies.

A point mass cannot undergo rotation, but only translatory motion. A rigid body with finite mass has both rotatory and translatory motion. The rotatory motion also can have three co-ordinates in space, like translatory motion ; Therefore a rigid body will have six degrees of freedom ; three due to translatory motion and three due to rotatory motion.

8.2.1 Monoatomic molecule

Since a monoatomic molecule consists of only a single atom of point mass it has three degrees of freedom of translatory motion along the three co-ordinate axes as shown in Fig. 8.3.

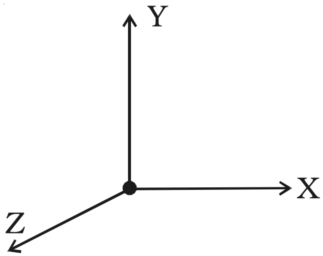


Fig. 8.3 Monoatomic molecule

Examples : molecules of rare gases like helium, argon, etc.

8.2.2 Diatomic molecule

The diatomic molecule can rotate about any axis at right angles to its own axis. Hence it has two degrees of freedom of rotational motion in addition to three degrees of freedom of translational motion along the three axes. So, a diatomic molecule has five degrees of freedom (Fig. 8.4). Examples : molecules of O_2, N_2, CO, Cl_2, etc.

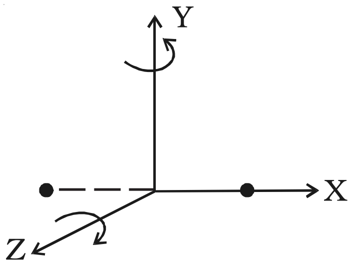


Fig. 8.4 Diatomic molecule

8.2.3 Triatomic molecule (Linear type)

In the case of triatomic molecule of linear type, the centre of mass lies at the central atom. It, therefore, behaves like a diamotic moelcule with three degrees of freedom of translation and two degrees of freedom of rotation, totally it has five degrees of freedom (Fig. 8.5). Examples : molecules of CO_2, CS_2, etc.

Fig. 8.5 Triatomic molecules (linear type)

8.2.4 Triatomic molecule (Non-linear type)

A triatomic non-linear molecule may rotate, about the three mutually perpendicular axes, as shown in Fig.8.6. Therefore, it possesses three degrees of freedom of rotation in addition to three degrees of freedom of translation along the three co-ordinate axes. Hence it has six degrees of freedom. Examples : molecules of H_2O, SO_2, etc.

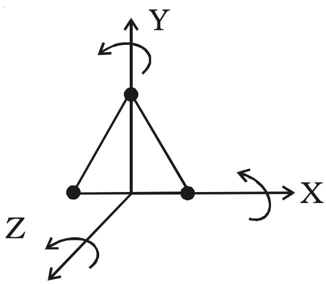


Fig. 8.6 Triatomic molecule

In all the above cases, only the translatory and rotatory motion of the molecules have been considered. The vibratory motion of the molecules has not been taken into consideration.

8.3 Law of equipartition of energy

Law of equipartition of energy states that for a dynamical system in thermal equilibrium the total energy of the system is shared equally by all the degrees of freedom. The energy associated with each degree of freedom per moelcule is $\frac{1}{2}$ kT, where k is the Boltzmann's constant.

Let us consider one mole of a monoatomic gas in thermal equilibrium at temperature T. Each molecule has 3 degrees of freedom due to translatory motion. According to kinetic theory of gases, the mean kinetic energy of a molecule is $\frac{3}{2}$ kT.

Let C_x, C_y and C_z be the components of RMS velocity of a molecule along the three axes. Then the average energy of a gas molecule is given by

$$\frac{1}{2} mC^2 = \frac{1}{2}mC_x^{\ 2} + \frac{1}{2}mC_y^{\ 2} + \frac{1}{2}mC_z^{\ 2}$$

$$\therefore \frac{1}{2}mC_x^{\ 2} + \frac{1}{2}mC_y^{\ 2} + \frac{1}{2}mC_z^{\ 2} = \frac{3}{2}kT$$

Since molecules move at random, the average kinetic energy corresponding to each degree of freedom is the same.

$$\therefore \frac{1}{2}mC_x^{\ 2} = \frac{1}{2}mC_y^{\ 2} = \frac{1}{2}mC_z^{\ 2}$$

$$\text{(i.e)} \qquad \frac{1}{2}mC_x^{\ 2} = \frac{1}{2}mC_y^{\ 2} = \frac{1}{2}mC_z^{\ 2} = \frac{1}{2}kT$$

$\therefore$ Mean kinetic energy per molecule per degree of freedom is $\frac{1}{2}$ kT.

8.4. Thermal equilibrium

Let us consider a system requiring a pair of independent co-ordinates X and Y for their complete description. If the values of X and Y remain unchanged so long as the external factors like temperature also remains the same, then the system is said to be in a state of thermal equilibrium.

Two systems A and B having their thermodynamic co-ordinates X and Y and X_1 and Y_1 respectively separated from each other, for example, by a wall, will have new and common co-ordinates X' and Y'

spontaneously, if the wall is removed. Now the two systems are said to be in thermal equilibrium with each other.

8.4.1 Zeroth law of thermodynamics

If two systems A and B are separately in thermal equilibrium with a third system C, then the three systems are in thermal equilibrium with each other. *Zeroth law of thermodynamics states that two systems which are individually in thermal equilibrium with a third one, are also in thermal equilibrium with each other.*

This Zeroth law was stated by Flower much later than both first and second laws of thermodynamics.

This law helps us to define temperature in a more rigorous manner.

8.4.2 Temperature

If we have a number of gaseous systems, whose different states are represented by their volumes and pressures V_1, V_2, V_3 ... and P_1, P_2, P_3... etc., in thermal equilibrium with one another, we will have $\phi_1 (P_1,V_1) = \phi_2 (P_2, V_2) = \phi_3 (P_3, V_3)$ and so on, where ϕ is a function of P and V. Hence, despite their different parameters of P and V, the numerical value of the these functions or the temperature of these systems is same.

Temperature may be defined as the particular property which determines whether a system is in thermal equilibrium or not with its neighbouring system when they are brought into contact.

8.5 Specific heat capacity

Specific heat capacity of a substance is defined as the quantity of heat required to raise the temperature of 1 kg of the substance through 1K. Its unit is J $kg^{-1}K^{-1}$.

Molar specific heat capacity of a gas

Molar specific heat capacity of a gas is defined as the quantity of heat required to raise the temperature of 1 mole of the gas through 1K. Its unit is J mol^{-1} K^{-1}.

Specific heat capacity of a gas may have any value between $-\infty$ and $+\infty$ depending upon the way in which heat energy is given.

Let m be the mass of a gas and C its specific heat capacity. Then $\Delta Q = m \times C \times \Delta T$ where ΔQ is the amount of heat absorbed and ΔT is the corresponding rise in temperature.

$$\text{(i.e)}\ C = \frac{\Delta Q}{m\,\Delta T}$$

Case (i)

If the gas is insulated from its surroundings and is suddenly compressed, it will be heated up and there is rise in temperature, even though no heat is supplied from outside

(i.e) $\Delta Q = 0$

$\therefore\ C = 0$

Case (ii)

If the gas is allowed to expand slowly, in order to keep the temperature constant, an amount of heat ΔQ is supplied from outside,

$$\text{then } C = \frac{\Delta Q}{m \times \Delta T} = \frac{\Delta Q}{0} = +\infty$$

($\because$ ΔQ is +ve as heat is supplied from outside)

Case (iii)

If the gas is compressed gradually and the heat generated ΔQ is conducted away so that temperature remains constant, then

$$C = \frac{\Delta Q}{m \times \Delta T} = \frac{-\Delta Q}{0} = -\infty$$

($\because$ ΔQ is -ve as heat is supplied by the system)

Thus we find that if the external conditions are not controlled, the value of the specific heat capacity of a gas may vary from $+\infty$ to $-\infty$

Hence, *in order to find the value of specific heat capacity of a gas, either the pressure or the volume of the gas should be kept constant.* Consequently a gas has two specific heat capacities (i) Specific heat capacity at constant volume (ii) Specific heat capacity at constant pressure.

Molar specific heat capacity of a gas at constant volume

Molar specific heat capacity of a gas at constant volume C_V is defined as the quantity of heat required to raise the temperature of one mole of a gas through 1 K, keeping its volume constant.

Molar specific heat capacity of a gas at constant pressure

Molar specific heat capacity of a gas at constant pressure C_p is defined as the quantity of heat to raise the temperature of one mole of a gas through 1 K keeping its pressure constant.

Specific heat capacity of monoatomic, diatomic and triatomic gases

Monoatomic gases like argon, helium etc. have three degrees of freedom.

We know, kinetic energy per molecule, per degree of freedom is $\frac{1}{2}$ kT.

$\therefore$ Kinetic energy per molecule with three degrees of freedom is $\frac{3}{2}$ kT.

Total kinetic energy of one mole of the monoatomic gas is given by $E = \frac{3}{2} kT \times N = \frac{3}{2} RT$, where N is the Avogadro number.

$$\therefore \quad \frac{dE}{dT} = \frac{3}{2}R$$

If dE is a small amount of heat required to raise the temperature of 1 mole of the gas at constant volume, through a temperature dT,

$$dE = 1 \times C_V \times dT$$

$$C_V = \frac{dE}{dT} = \frac{3}{2}R$$

As $R = 8.31$ J mol^{-1} K^{-1}

$$C_V = \frac{3}{2} \times 8.31 = 12.465 \text{ J mol}^{-1} \text{ K}^{-1}$$

Then $C_P - C_V = R$

$$C_P = C_V + R$$

$$= \frac{3}{2}R + R = \frac{5}{2} R = \frac{5}{2} \times 8.31$$

$$\therefore \quad C_p = 20.775 \text{ J mol}^{-1} \text{ K}^{-1}$$

In diatomic gases like hydrogen, oxygen, nitrogen etc., a molecule has five degrees of freedom. Hence the total energy associated with one mole of diatomic gas is

$$E = 5 \times \frac{1}{2} kT \times N = \frac{5}{2} RT$$

$$\text{Also, } C_v = \frac{dE}{dT} = \frac{d}{dT}\left(\frac{5}{2}\text{RT}\right) = \frac{5}{2}R$$

$$C_v = \frac{5}{2} \times 8.31 = 20.775 \text{ J mol}^{-1} \text{ K}^{-1}$$

$$\text{But } C_p = C_v + R$$

$$= \frac{5}{2}R + R = \frac{7}{2}R$$

$$C_p = \frac{7}{2} \times 8.31$$

$$= 29.085 \text{ J mol}^{-1} \text{ K}^{-1}$$

similarly, Cp and Cv can be calculated for triatomic gases.

Internal energy

Internal energy U of a system is the energy possessed by the system due to molecular motion and molecular configuration. The internal kinetic energy U_K of the molecules is due to the molecular motion and the internal potential energy U_P is due to molecular configuration. Thus

$$U = U_K + U_P$$

It depends only on the initial and final states of the system. In case of an ideal gas, it is assumed that the intermolecular forces are zero. Therefore, no work is done, although there is change in the intermolecular distance. Thus $U_P = O$. Hence, internal energy of an ideal gas has only internal kinetic energy, which depends only on the temperature of the gas.

In a real gas, intermolecular forces are not zero. Therefore, a definite amount of work has to be done in changing the distance between the molecules. Thus the internal energy of a real gas is the sum of internal kinetic energy and internal potential energy. Hence, it would depend upon both the temperature and the volume of the gas.

8.6 First law of thermodynamics

Let us consider a gas inside a cylinder fitted with a movable frictionless piston. The walls of the cylinder are made up of non-conducting material and the bottom is made up of conducting material (Fig. 8.7).

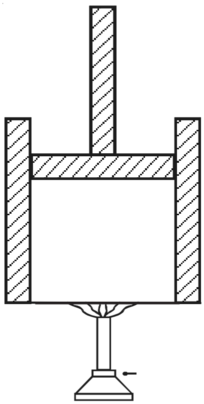


Fig. 8.7 First Law of thermodynamics

Let the bottom of the cylinder be brought in contact with a hot body like burner. The entire heat energy given to the gas is not converted into work. A part of the heat energy is used up in increasing the temperature of the gas (i.e) in increasing its internal energy and the remaining energy is used up in pushing the piston upwards (i.e.) in doing work.

If ΔQ is the heat energy supplied to the gas, U_1 and U_2 are initial and final internal energies and ΔW is the work done by the system, then

$$\Delta Q = \Delta W + (U_2 - U_1)$$

$$\Delta Q = \Delta W + \Delta U$$

where ΔU is the change in the internal energy of the system.

Hence, *the first law of thermodynamics states that the amount of heat energy supplied to a system is equal to the sum of the change in internal energy of the system and the work done by the system.* This law is in accordance with the law of conservation of energy.

8.7 Relation between C_p and C_v (Meyer's relation)

Let us consider one mole of an ideal gas enclosed in a cylinder provided with a frictionless piston of area *A*. Let *P*, *V* and *T* be the pressure, volume and absolute temperature of gas respectively (Fig. 8.8).

A quantity of heat dQ is supplied to the gas. To keep the volume of the gas constant, a small weight is placed over the piston. The pressure and the temperature of the gas increase to $P + dP$ and $T + dT$ respectively. This heat energy dQ is used to increase the internal energy dU of the gas. But the gas does not do any work ($dW = 0$).

$$\therefore\ dQ = dU = 1 \times C_v \times dT \qquad \text{... (1)}$$

The additional weight is now removed from the piston. The piston now moves upwards through a distance dx, such that the pressure of

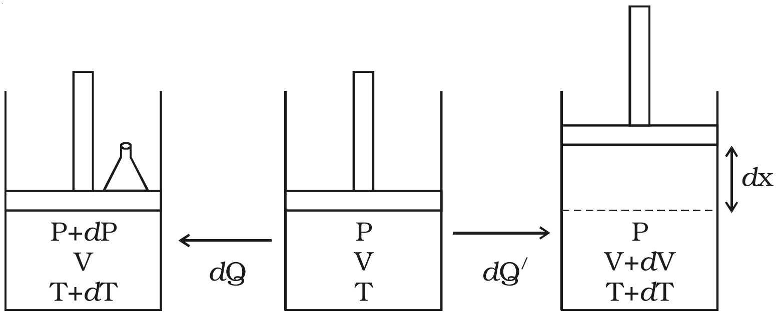


Fig. 8.8 Meyer's relation

the enclosed gas is equal to the atmospheric pressure P. The temperature of the gas decreases due to the expansion of the gas.

Now a quantity of heat dQ' is supplied to the gas till its temperature becomes $T + dT$. This heat energy is not only used to increase the internal energy dU of the gas but also to do external work dW in moving the piston upwards.

$$\therefore \; dQ' = dU + dW$$

Since the expansion takes place at constant pressure,

$$dQ' = C_p dT$$

$$\therefore \; C_p dT = C_v dT + dW \qquad \dots (2)$$

Work done, dW = force × distance

$$= P \times A \times dx$$

$$dW = P\,dV \; (\text{since } A \times dx = dV, \text{ change in volume})$$

$$\therefore \; C_p dT = C_v dT + P\,dV \qquad \dots (3)$$

The equation of state of an ideal gas is

$$PV = RT$$

Differentiating both the sides

$$PdV = RdT \qquad \dots (4)$$

Substituting equation (4) in (3),

$$C_p dT = C_v dT + RdT$$

$$C_p = C_v + R$$

$$\therefore \quad C_p - C_v = R$$

This equation is known as Meyer's relation

8.8 Indicator diagram (P-V diagram)

A curve showing variation of volume of a substance taken along the X-axis and the variation of pressure taken along Y-axis is called an indicator diagram or P-V diagram. The shape of the indicator diagram shall depend on the nature of the thermodynamical process the system undergoes.

Let us consider one mole of an ideal gas enclosed in a cylinder fitted with a perfectly frictionless piston. Let P_1, V_1 and T be the initial state of the gas. If dV is an infinitesimally small increase in volume of the gas during which the pressure P is assumed to be constant, then small amount of workdone by the gas is $dW = PdV$

In the indicator diagram dW = area $a_1b_1c_1d_1$

$\therefore$ The total workdone by the gas during expansion from V_1 to V_2 is

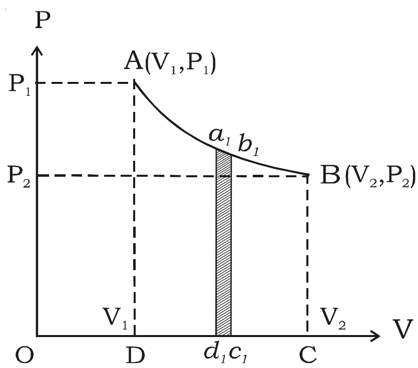


Fig. 8.9 Indicator diagram

$$W = \int_{V_1}^{V_2} PdV = \text{Area ABCD, in the}$$

indicator diagram.

Hence, in an indicator diagram the area under the curve represents the work done (Fig. 8.9).

8.8.1 Isothermal process

When a gas undergoes expansion or compression at constant temperature, the process is called isothermal process.

Let us consider a gas in a cylinder provided with a frictionless piston. The cylinder and the piston are made up of conducting material. If the piston is pushed down slowly, the heat energy produced will be quickly transmitted to the surroundings. Hence, the temperature remains constant but the pressure of the gas increases and its volume decreases.

The equation for an isothermal process is PV = constant.

If a graph is drawn between P and V, keeping temperature constant, we get a curve called an isothermal curve. Isotherms for three different

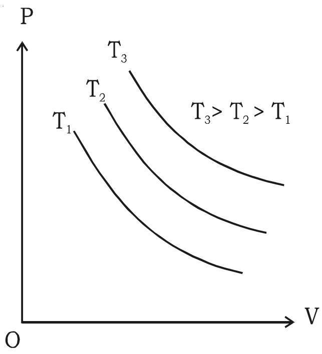


Fig. 8.10 Isothermal process

temperatures T_1, T_2 and T_3 are shown in the Fig. 8.10. The curve moves away from the origin at higher temperatures.

During an isothermal change, the specific heat capacity of the gas is infinite.

(i.e) $C = \frac{\Delta Q}{m\Delta T} = \infty \qquad (\because \Delta T = 0)$

(e.g) Melting of ice at its melting point and vapourisation of water at its boiling point.

8.8.2 Workdone in an isothermal expansion

Consider one mole of an ideal gas enclosed in a cylinder with perfectly conducting walls and fitted with a perfectly frictionless and conducting piston. Let P_1, V_1 and T be the initial pressure, volume and temperature of the gas. Let the gas expand to a volume V_2 when pressure reduces to P_2, at constant temperature T. At any instant during expansion let the pressure of the gas be P. If A is the area of cross section of the piston, then force $F = P \times A$.

Let us assume that the pressure of the gas remains constant during an infinitesimally small outward displacement dx of the piston. Work done

$$dW = Fdx = PAdx = PdV$$

Total work done by the gas in expansion from initial volume V_1 to final volume V_2 is

$$W = \int_{V_1}^{V_2} P\, dV$$

We know, $PV = RT$, $P = \frac{RT}{V}$

$$\therefore W = \int_{V_1}^{V_2} \frac{RT}{V}\, dV = RT \int_{V_1}^{V_2} \frac{1}{V} dV$$

$$W = RT\, [\log_e V]_{V_1}^{V_2}$$

$$W = RT\left[\log_e V_2 - \log_e V_1\right]$$

$$= RT\ log_e \frac{V_2}{V_1}$$

$$W = 2.3026\ RT\ log_{10} \frac{V_2}{V_1}$$

This is the equation for the workdone during an isothermal process.

8.8.3 Adiabatic process

In Greek, adiabatic means "nothing passes through". *The process in which pressure, volume and temperature of a system change in such a manner that during the change no heat enters or leaves the system is called adiabatic process.* Thus in adiabatic process, the total heat of the system remains constant.

Let us consider a gas in a perfectly thermally insulated cylinder fitted with a piston. If the gas is compressed suddenly by moving the piston downward, heat is produced and hence the temperature of the gas will increase. Such a process is adiabatic compression.

If the gas is suddenly expanded by moving the piston outward, energy required to drive the piston is drawn from the internal energy of the gas, causing fall in temperature. This fall in temperature is not compensated by drawing heat from the surroundings. This is adiabatic expansion.

Both the compression and expansion should be sudden, so that there is no time for the exchange of heat. Hence, *in an adiabatic process always there is change in temperature.*

Expansion of steam in the cylinder of a steam engine, expansion of hot gases in internal combustion engine, bursting of a cycle tube or car tube, propagation of sound waves in a gas are adiabatic processes.

The adiabatic relation between P and V for a gas, is

$PV^{\gamma} = k$, a constant ... (1)

where $\gamma = \dfrac{\text{specific heat capacity of the gas at constant pressure}}{\text{specific heat capacity of the gas at constant volume}}$

From standard gas equation,

$$PV = RT$$

$$P = \frac{RT}{V}$$

substituting the value P in (1)

$$\frac{RT}{V} V^{\gamma} = \text{constant}$$

$$T.V^{\gamma-1} = \text{constant}$$

In an adiabatic process Q =constant

$$\therefore \Delta Q = 0$$

$\therefore$ specific heat capacity $C = \dfrac{\Delta Q}{m\Delta T}$

$$\therefore C = 0$$

8.8.4 Work done in an adiabatic expansion

Consider one mole of an ideal gas enclosed in a cylinder with perfectly non conducting walls and fitted with a perfectly frictionless, non conducting piston.

Let P_1, V_1 *and* T_1 be the initial pressure, volume and temperature of the gas. If A is the area of cross section of the piston, then force exerted by the gas on the piston is

$F = P \times A$, where P is pressure of the gas at any instant during expansion. If we assume that pressure of the gas remains constant during an infinitesimally small outward displacement dx of the piston,

then work done $dW = F \times dx = P \times A\ dx$

$$dW = P\ dV$$

Total work done by the gas in adiabatic expansion from volume V_1 to V_2 is

$$W = \int_{V_1}^{V_2} P\ dV$$

But PV^{γ} = constant (k) for adiabatic process

where $\gamma = \dfrac{C_P}{C_V}$

$$\therefore \quad W = \int_{V_1}^{V_2} k.V^{-\gamma}\ dV = k\left[\frac{V^{1-\gamma}}{1-\gamma}\right]_{V_1}^{V_2} \quad \left(\because P = \frac{k}{V^{\gamma}}\right)$$

$$W = \frac{k}{1-\gamma}\left[V_2^{1-\gamma} - V_1^{1-\gamma}\right]$$

$$W = \frac{1}{1-\gamma}\left[kV_2^{1-\gamma} - kV_1^{1-\gamma}\right] \quad \text{... (1)}$$

$$but,\ P_2V_2^{\gamma} = P_1V_1^{\gamma} = k \quad \text{... (2)}$$

Substituting the value of k in (1)

$$\therefore\ W = \frac{1}{1-\gamma}\left[P_2V_2^{\gamma}\,.\,V_2^{1-\gamma} - P_1\,V_1^{\gamma}\,V_1^{1-\gamma}\right]$$

$$W = \frac{1}{1-\gamma}\left[P_2\,V_2 - P_1V_1\right] \quad \text{... (3)}$$

If T_2 is the final temperature of the gas in adiabatic expansion, then

$$P_1V_1 = RT_1,\ P_2V_2 = RT_2$$

Substituting in (3)

$$W = \frac{1}{1-\gamma}\left[RT_2 - RT_1\right]$$

$$W = \frac{R}{1-\gamma}\left[T_2 - T_1\right] \quad \text{... (4)}$$

This is the equation for the work done during adiabatic process.

8.9 Reversible and irreversible processes

8.9.1 Reversible process

A thermodynamic process is said to be reversible when (i) the various stages of an operation to which it is subjected can be reversed in the opposite direction and in the reverse order and (ii) in every part of the process, the amount of energy transferred in the form of heat or work is the same in magnitude in either direction. At every stage of the process there is no loss of energy due to friction, inelasticity, resistance, viscosity etc. The heat losses to the surroundings by conduction, convection or radiation are also zero.

Condition for reversible process

(i) The process must be infinitely slow.

(ii) The system should remain in thermal equilibrium (i.e) system and surrounding should remain at the same temperature.

Examples

(a) Let a gas be compressed isothermally so that heat generated is conducted away to the surrounding. When it is allowed to expand in the same small equal steps, the temperature falls but the system takes up the heat from the surrounding and maintains its temperature.

(b) Electrolysis can be regarded as reversible process, provided there is no internal resistance.

8.9.2 Irreversible process

An irreversible process is one which cannot be reversed back. Examples : diffusion of gases and liquids, passage of electric current through a wire, and heat energy lost due to friction. As an irreversible process is generally a very rapid one, temperature adjustments are not possible. Most of the chemical reactions are irreversible.

8.10 Second law of thermodynamics

The first law of thermodynamics is a general statement of equivalence between work and heat. The second law of thermodynamics enables us to know whether a process which is allowed by first law of thermodynamics can actually occur or not. The second law of thermodynamics tells about the extent and direction of energy transformation.

Different scientists have stated this law in different ways to bring out its salient features.

(i) Kelvin's statement

Kelvin's statement of second law is based on his experience about the performance of heat engine.

It is impossible to obtain a continuous supply of work from a body by cooling it to a temperature below the coldest of its surroundings.

(ii) Clausius statement

It is impossible for a self acting machine unaided by any external *agency to transfer heat from a body at a lower temperature to another body at a higher temperature.*

(iii) Kelvin - Planck's statement

It is impossible to construct a heat engine operating in a cycle, that will extract heat from a reservoir and perform an equivalent amount of work.

8.11 Carnot engine

Heat engine is a device which converts heat energy into mechanical energy.

In the year 1824, Carnot devised an ideal cycle of operation for a heat engine. The machine used for realising this ideal cycle of operation is called an ideal heat engine or carnot heat engine.

The essential parts of a Carnot engine are shown in Fig. 8.11

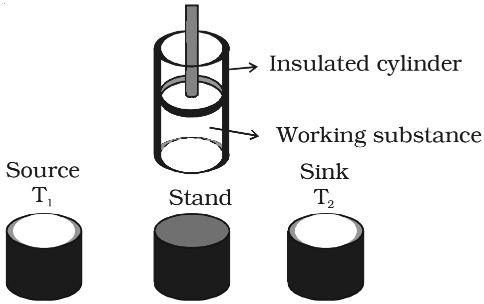


Fig. 8.11 Carnot engine

(i) Source

It is a hot body which is kept at a constant temperature T_1. It has infinite thermal capacity. Any amount of heat can be drawn from it at a constant temperature T_1 (i.e) its temperature will remain the same even after drawing any amount of heat from it.

(ii) Sink

It is a cold body which is kept at a constant lower temperature T_2. Its thermal capacity is also infinite that any amount of heat added to it will not increase its temperature.

(iii) Cylinder

Cylinder is made up of non-conducting walls and conducting bottom. A perfect gas is used as a working substance. The cylinder is fitted with a perfectly non-conducting and frcitionless piston.

(iv) Insulating stand

It is made up of non conducting material so as to perform adiabatic operations.

Working : The Carnot engine has the following four stages of operations.

1. Isothermal expansion 2. Adiabatic expansion 3. Isothermal compression 4. Adiabatic compression.

Isothermal expansion

Let us consider one mole of an ideal gas enclosed in the cylinder. Let V_1, P_1 be the initial volume and pressure of the gas respectively. The initial state of the gas is represented by the point *A* on the *P–V* diagram. The cylinder is placed over the source which is at the temperature T_1.

The piston is allowed to move slowly outwards, so that the gas expands. Heat is gained from the source and the process is isothermal at constant temperature T_1. In this process the volume of the gas changes from V_1 to V_2 and the pressure changes from P_1 to P_2. This process is represented by *AB* in the indicator diagram (Fig. 8.12). During this process, the quantity of heat absorbed from the source is Q_1 and W_1 is the corresponding amount of work done by the gas.

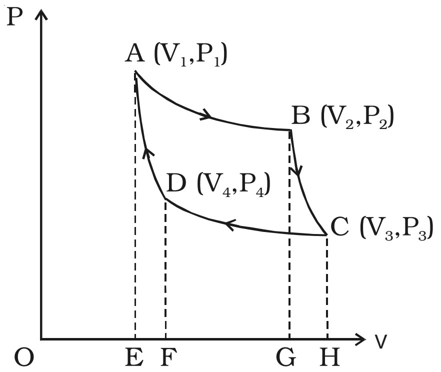


Fig. 8.12 Carnot cycle

$$\therefore Q_1 = W_1 = \int_{V_1}^{V_2} PdV = RT_1 \log_e \left(\frac{V_2}{V_1}\right)$$

= area ABGEA ...(1)

Adiabatic expansion

The cylinder is taken from the source and is placed on the insulting stand and the piston is moved further so that the volume of the gas changes from V_2 to V_3 and the pressure changes from P_2 to P_3. This adiabatic expansion is represented by BC. Since the gas is thermally insulated from all sides no heat can be gained from the surroundings. The temperature of the gas falls from T_1 to T_2.

Let W_2 be the work done by the gas in expanding adiabatically.

$$\therefore\ W_2 = \int_{V_2}^{V_3} PdV = \frac{R}{\gamma - 1}(T_1 - T_2) = \text{Area BCHGB} \qquad ...(2)$$

Isothermal compression

The cylinder is now placed on the sink at a temperature T_2. The piston is moved slowly downward to compress the gas isothermally. This is represented by CD. Let (V_4, P_4) be the volume and pressure corresponding to the point D. Since the base of the cylinder is conducting the heat produced during compression will pass to the sink so that, the temperature of the gas remains constant at T_2. Let Q_2 be the amount of heat rejected to the sink and W_3 be the amount of work done on the gas in compressing it isothermally.

$$Q_2 = W_3 = \int_{V_3}^{V_4} -P\ dV = -RT_2 \log_e\left(\frac{V_4}{V_3}\right) = -\text{ area CDFHC} \qquad ...(3)$$

The negative sign indicates that work is done on the working substance.

$$\therefore\ Q_2 = RT_2 \log_e\left(\frac{V_3}{V_4}\right)$$

Adiabatic compression

The cylinder is now placed on the insulating stand and the piston is further moved down in such a way that the gas is compressed adiabatically to its initial volume V_1 and pressure P_1. As the gas is insulated from all sides heat produced raises the temperature of the gas to T_1. This change is adiabatic and is represented by DA. Let W_4 be the work done on the gas in compressing it adiabatically from a state $D\ (V_4, P_4)$ to the initial state $A\ (V_1, P_1)$.

$$\therefore W_4 = \int_{V_4}^{V_1} -P\,dV = \frac{-R}{\gamma - 1}(T_2 - T_1)$$

The negative sign indicates that work is done on the working substance.

$$\therefore W_4 = \frac{R}{\gamma - 1}(T_1 - T_2) = \text{Area DAEFD} \qquad ...(4)$$

Work done by the engine per cycle

Total work done by the gas during one cycle of operation is $(W_1 + W_2)$.

Total work done on the gas during one cycle of operation is $(W_3 + W_4)$.

$\therefore$ Net work done by the gas in a complete cycle

$$W = W_1 + W_2 - (W_3 + W_4)$$

But $W_2 = W_4$

$$\therefore W = W_1 - W_3$$

$$W = Q_1 - Q_2$$

Also, W = Area ABGEA + Area BCHGB - Area CDFHC - Area DAEFD

(i.e) W = Area ABCDA

Hence in Carnot heat engine, *net work done by the gas per cycle is numerically equal to the area of the loop representing the cycle.*

Efficiency of Carnot's engine

$$\eta = \frac{\text{Heat converted into work}}{\text{Heat drawn from the source}} = \frac{Q_1 - Q_2}{Q_1}$$

$$\eta = 1 - \frac{Q_2}{Q_1}$$

But $$\frac{Q_1}{Q_2} = \frac{W_1}{W_3} = \frac{RT_1 \log\left(V_2/V_1\right)}{RT_2 \log\left(V_3/V_4\right)}$$

$$= \frac{T_1 \log\left(V_2/V_1\right)}{T_2 \log\left(V_3/V_4\right)} \qquad ...(5)$$

Since B and C lie on the same adiabatic curve BC

$T_1V_2^{\gamma-1} = T_2V_3^{\gamma-1}$ ($\because$ $TV^{\gamma-1}$ = constant) where $\gamma = \frac{C_p}{C_v}$

$$\therefore \frac{T_1}{T_2} = \frac{V_3^{\gamma-1}}{V_2^{\gamma-1}} \qquad ...(6)$$

Similarly D & A lie on the same adiabatic curve DA

$$\therefore T_1V_1^{\gamma-1} = T_2V_4^{\gamma-1}$$

$$\frac{T_1}{T_2} = \frac{V_4^{\gamma-1}}{V_1^{\gamma-1}} \qquad ...(7)$$

From (6) & (7) $\frac{V_3^{\gamma-1}}{V_2^{\gamma-1}} = \frac{V_4^{\gamma-1}}{V_1^{\gamma-1}}$

$$\frac{V_3}{V_2} = \frac{V_4}{V_1} \text{ (or) } \frac{V_2}{V_1} = \frac{V_3}{V_4} \qquad ...(8)$$

substituting equation (8) in equation (5)

$$\frac{Q_1}{Q_2} = \frac{T_1}{T_2} \frac{\log\left(V_3/V_4\right)}{\log\left(V_3/V_4\right)}$$

(i.e) $\frac{Q_2}{Q_1} = \frac{T_2}{T_1}$

$\therefore$ We have $\eta = 1 - \frac{Q_2}{Q_1} = 1 - \frac{T_2}{T_1}$

$$\text{or} \quad \eta = \frac{T_1 - T_2}{T_1} \qquad ...(9)$$

Inferences

Efficiency of Carnot's cycle is independent of the working substance, but depends upon the temperatures of heat source and sink.

Efficiency of Carnot's cycle will be 100% if $T_1 = \infty$ *or* $T_2 = 0$ K. As neither the temperature of heat source can be made infinite, nor the temperature of the sink can be made 0 K, the inference is that the

Carnot heat engine working on the reversible cycle cannot have 100% efficiency.

8.12 Refrigerator

A refrigerator is a cooling device. *An ideal refrigerator can be regarded as Carnot's heat engine working in the reverse direction. Therefore, it is also called a heat pump.* In a refrigerator the working substance would absorb certain quantity of heat from the sink at lower temperature and reject a large amount of heat to the source at a higher temperature with the help of an external agency like an electric motor (Fig. 8.13).

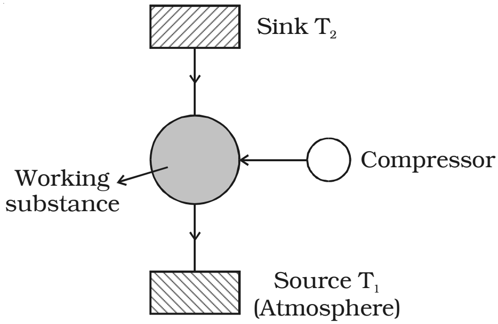


Fig. 8.13 Refrigerator

In an actual refrigerator vapours of freon (dichloro difluoro methane CCl_2F_2) act as the working substance. Things kept inside the refrigerator act as a sink at a lower temperature T_2. A certain amount of work W is performed by the compressor (operated by an electric motor) on the working substance. Therefore, it absorbs heat energy Q_2 from the sink and rejects Q_1 amount of heat energy to the source (atmosphere) at a temperature T_1.

Since this is a reversible cyclic process, the change in the internal energy of the working substance is zero (i.e) $dU = 0$

According to the first law of thermodynamics,

$$dQ = dU + dW$$

But $$dQ = Q_2 - Q_1$$

$$dW = -W \quad \text{and} \quad dU = 0$$

$$\therefore \quad dQ = Q_2 - Q_1 = -W$$

Negative sign with W represents work is done on the system

(i.e) $W = Q_1 - Q_2$

Coefficient of performance

Coefficient of performance (COP) is defined as the ratio of quantity of heat Q_2 removed per cycle from the contents of the refrigerator to the energy spent per cycle W to remove this heat.

$$\text{(i.e) COP} = \frac{Q_2}{W} = \frac{Q_2}{Q_1 - Q_2}$$

$$\text{(i.e) COP} = \frac{T_2}{T_1 - T_2} \quad \ldots (1)$$

The efficiency of the heat engine is

$$\eta = 1 - \frac{T_2}{T_1}; \qquad 1 - \eta = \frac{T_2}{T_1}$$

$$\frac{1-\eta}{\eta} = \frac{T_2}{T_1} \times \frac{T_1}{T_1 - T_2}$$

$$\text{(i.e)} \quad \frac{1-\eta}{\eta} = \frac{T_2}{T_1 - T_2} \quad \ldots (2)$$

From equations (1) and (2)

$$\text{COP} = \frac{1-\eta}{\eta} \quad \ldots (3)$$

Inferences

(i) Equation (1) shows that smaller the value of $(T_1 - T_2)$ greater is the value of COP. (i.e.) smaller is the difference in temperature between atmosphere and the things to be cooled, higher is the COP.

(ii) As the refrigerator works, T_2 goes on decreasing due to the formation of ice. T_1 is almost steady. Hence COP decreases. When the refrigerator is defrosted, T_2 increases.

Therefore defrosting is essential for better working of the refrigerator.

8.13 Transfer of heat

There are three ways in which heat energy may get transferred from one place to another. These are conduction, convection and radiation.

8.13.1 Conduction

Heat is transmitted through the solids by the process of conduction only. When one end of the solid is heated, the atoms or molecules of the solid at the hotter end becomes more strongly agitated and start vibrating with greater amplitude. The disturbance is transferred to the neighbouring molecules.

Applications

(i) The houses of Eskimos are made up of double walled blocks of ice. Air enclosed in between the double walls prevents transmission of heat from the house to the coldest surroundings.

(ii) Birds often swell their feathers in winter to enclose air between their body and the feathers. Air prevents the loss of heat from the body of the bird to the cold surroundings.

(iii) Ice is packed in gunny bags or sawdust because, air trapped in the saw dust prevents the transfer of heat from the surroundings to the ice. Hence ice does not melt.

Coefficient of thermal conductivity

Let us consider a metallic bar of uniform cross section A whose one end is heated. After sometime each section of the bar attains constant temperature but it is different at different sections. This is called steady state. In this state there is no further absorption of heat.

If Δx is the distance between the two sections with a difference in temperature of ΔT and ΔQ is the amount of heat conducted in a time Δt, then it is found that the rate of conduction of heat $\frac{\Delta Q}{\Delta t}$ is

(i) directly proportional to the area of cross section (A)

(ii) directly proportional to the temperature difference between the two sections (ΔT)

(iii) inversely proportional to the distance between the two sections (Δx).

(i.e) $$\frac{\Delta Q}{\Delta t} \propto A \frac{\Delta T}{\Delta x}$$

$$\frac{\Delta Q}{\Delta t} = KA \frac{\Delta T}{\Delta x}$$

where K is a constant of proportionality called co-efficient of thermal conductivity of the metal.

$\frac{\Delta T}{\Delta x}$ is called temperature gradient

If $A = 1\ m^2$, and $\frac{\Delta T}{\Delta x}$ = unit temperature gradient

then, $\frac{\Delta Q}{\Delta t} = K \times 1 \times 1$

or $K = \frac{\Delta Q}{\Delta t}$

Coefficient of thermal conductivity of the material of a solid is equal to the rate of flow of heat per unit area per unit temperature gradient across the solid. Its unit is $W\ m^{-1}\ K^{-1}$.

8.13.2 Convection

It is a phenomenon of transfer of heat in a fluid with the actual movement of the particles of the fluid.

When a fluid is heated, the hot part expands and becomes less dense. It rises and upper colder part replaces it. This again gets heated, rises up replaced by the colder part of the fluid. This process goes on. This mode of heat transfer is different from conduction where energy transfer takes place without the actual movement of the molecules.

Application

It plays an important role in ventilation and in heating and cooling system of the houses.

8.13.3 Radiation

It is the phenomenon of transfer of heat without any material medium. Such a process of heat transfer in which no material medium takes part is known as radiation.

Thermal radiation

The energy emitted by a body in the form of radiation on account of its temperature is called thermal radiation.

It depends on,

(i) temperature of the body,

(ii) nature of the radiating body

The wavelength of thermal radiation ranges from 8×10^{-7} *m* to 4×10^{-4} *m*. They belong to infra-red region of the electromagnetic spectrum.

Properties of thermal radiations

1. Thermal radiations can travel through vacuum.
2. They travel along straight lines with the speed of light.
3. They can be reflected and refracted. They exhibit the phenomenon of interference and diffraction.
4. They do not heat the intervening medium through which they pass.
5. They obey inverse square law.

Absorptive and Emissive power

Absorptive power

Absorptive power of a body for a given wavelength and temperature is defined *as the ratio of the radiant energy absorbed per unit area per unit time to the total energy incident on it per unit area per unit time.*

It is denoted by a_λ.

Emissive power

Emissive power of a body at a given temperature is *the amount of energy emitted per unit time per unit area of the surface for a given wavelength*. It is denoted by e_λ. Its unit is $W\ m^{-2}$.

8.14 Perfect black body

A perfect black body is the one which absorbs completely heat radiations of all wavelengths which fall on it and emits heat radiations of all wavelengths when heated. Since a perfect black body neither reflects nor transmits any radiation, the absorptive power of a perfectly black body is unity.

8.14.1 Fery's black body

Fery's black body consists of a double walled hollow sphere having a small opening O on one side and a conical projection P just opposite to it (Fig. 8.14). Its inner surface is coated with lamp black. Any radiation entering the body through the opening O suffers multiple reflections at its innerwall and about 97% of it is absorbed by lamp black at each reflection. Therefore, after a few reflections almost entire radiation is absorbed. The projection helps in avoiding any direct reflections which even otherwise is not possible because of the small opening O. When this body is placed in a bath at fixed temperature, the heat radiations come out of the hole. The opening O thus acts as a black body radiator.

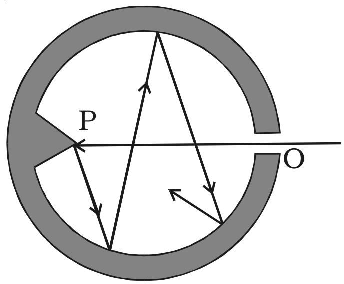


Fig. 8.14 Fery's black body

8.14.2 Prevost's theory of heat exchanges

Prevost applied the idea of thermal equilibrium to radiation. According to him the rate at which a body radiates or absorbs heat depends on the nature of its surface, its temperature and the temperature of the surroundings. The total amount of heat radiated by a body increases as its temperature rises. A body at a higher temperature radiates more heat energy to the surroundings than it receives from the surroundings. That is why we feel warm when we stand before the furnace.

Similarly a body at a lower temperature receives more heat energy than it loses to the surroundings. That is why we feel cold when we stand before an ice block.

Thus the rise or fall of temperature is due to the exchange of heat radiation. When the temperature of the body is the same as that of surroundings, the exchanges of heat do not stop. In such a case, the amount of heat energy radiated by the body is equal to the amount of heat energy absorbed by it.

A body will stop emitting radiation only when it is at absolute zero. (i.e) 0 K or -273^{o} C. At this temperature the kinetic energy of the molecule is zero.

Therefore, Prevost theory states that all bodies emit thermal radiation at all temperatures above absolute zero, irrespective of the nature of the surroundings.

8.14.3 Kirchoff's Law

According to this law, *the ratio of emissive power to the absorptive power corresponding to a particular wavelength and at a given temperature is always a constant for all bodies.* This constant is equal to the emissive power of a perfectly black body at the same temperature and the same wavelength. Thus, if e_λ is the emissive power of a body corresponding to a wavelength λ at any given temperature, a_λ is the absorptive power of the body corresponding to the same wavelength at the same temperature and E_λ is the emissive power of a perfectly black body corresponding to the same wavelength and the same temperature, then according to Kirchoff's law

$$\frac{e_\lambda}{a_\lambda} = \text{constant} = E_\lambda$$

From the above equation it is evident that if a_λ is large, then e_λ will also be large (i.e) if a body absorbs radiation of certain wavelength strongly then it will also strongly emit the radiation of same wavelength. In other words, good absorbers of heat are good emitters also.

Applications of Kirchoff's law

(i) The silvered surface of a thermos flask is a bad absorber as well as a bad radiator. Hence, ice inside the flask does not melt quickly and hot liquids inside the flask do not cool quickly.

(ii) Sodium vapours on heating, emit two bright yellow lines. These are called D_1 and D_2 lines of sodium. When continuous white light from carbon arc passes through sodium vapour at low temperature, the continuous spectrum is absorbed at two places corresponding to the wavelengths of D_1 and D_2 lines and appear as dark lines. This is in accordance with Kirchoff's law.

8.14.4 Wien's displacement law

Wien's displacement law *states that the wavelength of the radiation corresponding to the maximum energy (λ_m) decreases as the temperature T of the body increases.*

(i.e) $\lambda_m T = b$ where b is called Wien's constant.

Its value is 2.898×10^{-3} m K

8.14.5 Stefan's law

Stefan's law states that *the total amount of heat energy radiated per second per unit area of a perfect black body is directly proportional to the fourth power of its absolute temperature.*

(i.e) $E \alpha T^4$ or $E = \sigma T^4$

where σ is called the Stefan's constant. Its value is 5.67×10^{-8} W m^{-2} K^{-4}.

It is also called Stefan - Boltzmann law, as Boltzmann gave a theoretical proof of the result given by Stefan.

8.14.6 Newton's law of cooling

Newton's law of cooling states that *the rate of cooling of a body is directly proportional to the temperature difference between the body and the surroundings.*

The law holds good only for a small difference of temperature. Loss of heat by radiation depends on the nature of the surface and the area of the exposed surface.

Experimental verification of Newton's law of cooling

Let us consider a spherical calorimeter of mass m whose outer surface is blackened. It is filled with hot water of mass m_1. The calorimeter with a thermometer is suspended from a stand (Fig. 8.15).

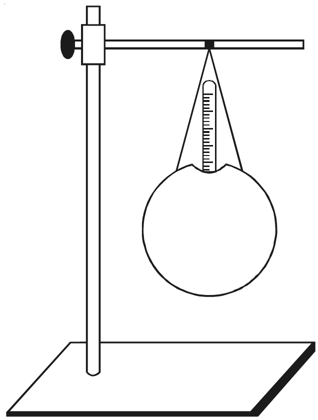


Fig. 8.15 Newton's law of cooling

The calorimeter and the hot water radiate heat energy to the surroundings. Using a stop clock, the temperature is noted for every 30 seconds interval of time till the temperature falls by about 20° C. The readings are entered in a tabular column.

If the temperature of the calorimeter and the water falls from T_1 to T_2 in t seconds, the quantity of heat energy lost by radiation $Q = (ms + m_1s_1)(T_1 - T_2)$, where s is the specific heat capacity of the material of the calorimeter and s_1 is the specific heat capacity of water.

Rate of cooling $= \frac{Heat\ energy\ lost}{time\ taken}$

$$\therefore \quad \frac{Q}{t} = \frac{(ms + m_1 s_1)(T_1 - T_2)}{t}$$

If the room temperature is T_o, the average excess temperature of the calorimeter over that of the surroundings is $\left(\frac{T_1 + T_2}{2} - T_o\right)$

According to Newton's Law of cooling, $\frac{Q}{t} \alpha \left(\frac{T_1 + T_2}{2} - T_o\right)$

$$\frac{(ms + m_1 s_1)(T_1 - T_2)}{t} \alpha \left(\frac{T_1 + T_2}{2} - T_o\right)$$

$$\therefore \quad \frac{(ms + m_1 s_1)(T_1 - T_2)}{t\left(\frac{T_1 + T_2}{2} - T_o\right)} = \text{constant}$$

The time for every 4^o fall in temperature is noted. The last column in the tabular column is found to be the same. This proves Newton's Law of cooling.

Table 8.1 Newton's law of cooling

Temperature range	Time t for every 4^o fall of temperature	Average excess of temperature $\left(\frac{T_1 + T_2}{2} - T_o\right)$	$\left(\frac{T_1 + T_2}{2} - T_o\right) t$

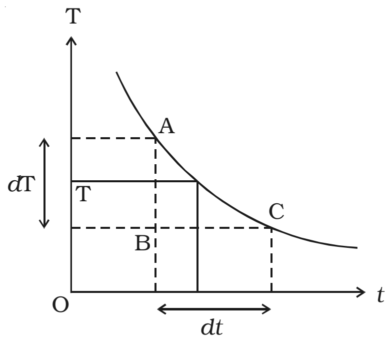


Fig. 8.16 Cooling curve

A cooling curve is drawn by taking time along X-axis and temperature along Y-axis (Fig. 8.16).

From the cooling curve, the rate of fall of temperature at T is $\frac{dT}{dt} = \frac{AB}{BC}$

The rate of cooling $\frac{dT}{dt}$ is found to be directly proportional to $(T - T_o)$. Hence Newton's law of cooling is verified.

8.15 Solar constant

The solar constant is the amount of radiant energy received per second per unit area by a perfect black body on the Earth with its surface perpendicular to the direction of radiation from the sun in the absence of atmosphere. It is denoted by S and its value is $1.388 \times 10^3\ W\ m^{-2}$. Surface temperature of the Sun can be calculated from solar constant.

Surface temperature of the Sun

The Sun is a perfect black body of radius r and surface temperature T. According to Stefan's law, the energy radiated by the Sun per second per unit area is equal to σT^4.

Where σ is Stefan's Constant.

Hence, the total energy radiated per second by the Sun will be given by

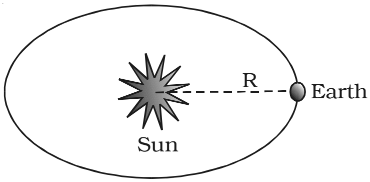


Fig. 8.17 Surface temperature of the Sun

$$E = \text{surface area of the Sun} \times \sigma T^4$$

$$E = 4\pi r^2\ \sigma T^4 \qquad \text{...(1)}$$

Let us imagine a sphere with Sun at the centre and the distance between the Sun and Earth R as radius (Fig. 8.17). The heat energy from the Sun will necessarily pass through this surface of the sphere.

If S is the solar constant, the amount of heat energy that falls on this sphere per unit time is $E = 4\pi R^2 S$ ...(2)

By definition, equations (1) & (2) are equal.

$$\therefore \quad 4\pi r^2 \sigma T^4. = 4\pi R^2 S$$

$$T^4 = \frac{R^2 S}{r^2 \sigma}$$

$$T = \left(\frac{R^2 S}{r^2 \sigma}\right)^{\frac{1}{4}}; \quad (i.e) \quad T = \left(\frac{R}{r}\right)^{\frac{1}{2}} \left(\frac{S}{\sigma}\right)^{\frac{1}{4}}$$

Knowing the values of R, r, S and σ the surface temperature of the Sun can be calculated.

8.15.1 Angstrom pyrheliometer

Pyrheliometer is an instrument used to measure the quantity of heat radiation and solar constant.

Pyrheliometer designed by Angstrom is the simplest and most accurate.

Angstrom's pyrheliometer consists of two identical strips S_1 and S_2 of area A. One junction of a thermocouple is connected to S_1 and the other junction is connected to S_2. A sensitive galvanometer is connected to the thermo couple.

Strip S_2 is connected to an external electrical circuit as shown in Fig.8.18. When both the strips S_1 and S_2 are shielded from the solar radiation, galvanometer shows no deflection as both the junctions are at the same temperature. Now strip S_1 is exposed to the solar radiation and S_2 is shielded with a cover M. As strip S_1 receives heat radiations from the sun, its temperature rises and hence the galvanometer shows deflection. Now current is allowed to pass through the strip S_2 and it is adjusted so that galvanometer shows no deflection. Now, the strips S_1 and S_2 are again at the same temperature.

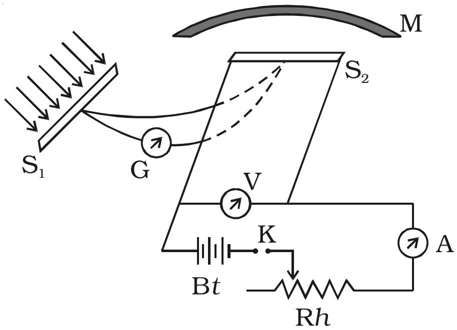


Fig. 8.18 Angstrom pyrheliometer

If the quantity of heat radiation that is incident on unit area in unit time on strip S_1 is Q and a its absorption co-efficient, then the amount of heat radiations absorbed by the strip S_1 in unit time is QAa.

Also, heat produced in unit time in the strip S_2 is given by VI, where V is the potential difference and I is the current flowing through it.

As heat absorbed = heat produced

$$QAa = VI \quad (or) \quad Q = \frac{VI}{Aa}$$

Knowing the values of V, I, A and a, Q can be calculated.

Solved Problems

8.1 At what temperature will the RMS velocity of a gas be tripled its value at NTP?

Solution : At NTP, $T_o = 273$ K

RMS velocity, $C = \sqrt{\frac{3RT_o}{M}}$

$$C = \sqrt{\frac{3R \times 273}{M}} \qquad \text{... (1)}$$

Suppose at the temperature T, the RMS velocity is tripled, then

$$3C = \sqrt{\frac{3RT}{M}} \qquad \text{... (2)}$$

Divide (2) by (1)

$$\frac{3C}{C} = \frac{\sqrt{\frac{3RT}{M}}}{\sqrt{\frac{3R \times 273}{M}}}$$

$$3 = \sqrt{\frac{T}{273}}$$

$$T = 273 \times 9 \qquad = 2457 \text{ K}$$

8.2 Calculate the number of degrees of freedom in 15 cm^3 of nitrogen at NTP.

Solution : *We know 22400 cm^3 of a gas at NTP contains 6.02×10^{23} molecules.*

$\therefore$ *The number of molecules in 15 cm^3 of N_2 at NTP*

$$n = \frac{15}{22400} \times 6.023 \times 10^{23} = 4.033 \times 10^{20}$$

The number degrees of freedom of a diatomic gas molecule at 273 K, is $f = 5$

$\therefore$ *Total degrees of freedom of 15 cm^3 of the gas = nf*

$\therefore$ *Total degrees of freedom = $4.033 \times 10^{20} \times 5 = 2.016 \times 10^{21}$*

8.3 A gas is a mixture of 2 moles of oxygen and 4 moles of argon at temperature T. Neglecting vibrational modes, show that the energy of the system is 11 RT where R is the universal gas constant.

Solution : *Since oxygen is a diatomic moleucle with 5 degrees of freedom, degrees of freedom of molecules in 2 moles of oxygen* $= f_1 = 2\,N \times 5 = 10\,N$

Since argon is a monatomic molecules degrees of freedom of molecules in 4 moles of argon $= f_2 = 4\,N \times 3 = 12\,N$

$\therefore$ *Total degrees of freedom of the mixture* $= f = f_1 + f_2 = 22\,N$

As per the principle of law of equipartition of energy, energy associated with each degree of freedom of a molecule $= \frac{1}{2}\,kT$

$\therefore$ *Total energy of the system* $= \frac{1}{2}\,kT \times 22\,N = 11\,RT$

8.4 Two carnot engines A and B are operating in series. The first one A receives heat at 600 K and rejects to a reservoir at temperature T. The second engine B receives the heat rejected by A and in turn rejects heat to a reservior at 150 K. Calculate the temperature T when (i) The work output of both the engines are equal, (ii) The efficiency of both the engines are equal.

Solution : *(i) When the work outputs are equal :*

For the first engine $W_1 = Q_1 - Q_2$

For the second engine $W_2 = Q_2 - Q_3$

Given (i.e) $W_1 = W_2$

$$Q_1 - Q_2 = Q_2 - Q_3$$

Divide by Q_2 *on both sides*

$$\frac{Q_1}{Q_2} - 1 = 1 - \frac{Q_3}{Q_2}$$

Also $\frac{Q_1}{Q_2} = \frac{600}{T}$

and $\frac{Q_2}{Q_3} = \frac{T}{150}$ $\qquad \left[\because \frac{Q_1}{Q_2} = \frac{T_1}{T_2}\right]$

$$\therefore \frac{600}{T} - 1 = 1 - \frac{150}{T}$$

$$\frac{600 - T}{T} = \frac{T - 150}{T}$$

$$\therefore T = 375 \ K$$

(ii) *When efficiencies are equal*

$$\eta_1 = 1 - \frac{Q_2}{Q_1} \quad \text{and} \quad \eta_2 = 1 - \frac{Q_3}{Q_2}$$

As $\eta_1 = \eta_2$

$$1 - \frac{Q_2}{Q_1} = 1 - \frac{Q_3}{Q_2}$$

$$1 - \frac{T}{600} = 1 - \frac{150}{T}$$

$$\frac{600 - T}{600} = \frac{T - 150}{T}$$

$$\frac{T}{600} = \frac{150}{T}$$

$$T^2 = 600 \times 150$$

$$\therefore \quad T = 300 \ K$$

8.5 A carnot engine whose low temperature reservoir is at 7° C has an efficiency of 50 %. It is desired to increase the efficiency to 70 %. By how many degrees should the temperature of the high temperature reservoir be increased?

Data : $\eta_1 = 50\ \% = 0.5$; $T_2 = 7 + 273 = 280K$; $\eta_2 = 70\% = 0.7$

Solution : $\eta_1 = 1 - \frac{T_2}{T_1}$; $0.5 = 1 - \frac{280}{T_1}$; $\therefore T_1 = 560\ K$

Let the temperature of the high temperature reservoir be T_1'

$$\eta_2 = 1 - \frac{T_2}{T_1'} \; ; 0.7 = 1 - \frac{280}{T_1'} \; ; \quad \therefore T_1' = 933.3\ K$$

$\therefore$ *The temperature of the reservoir should be increased by*

$933.3\ K - 560\ K = 373.3\ K$

8.6 A carnot engine is operated between two reservoirs at temperature 177° C and 77° C. If the engine receives 4200 J of heat energy from the source in each cycle, calculate the amount of heat rejected to the sink in each cycle. Calculate the efficiency and work done by the engine.

Data : $T_1 = 177^\circ C = 177 + 273 = 450\ K.$

$T_2 = 77^\circ C = 77 + 273 = 350\ K$

$Q_1 = 4200\ J \quad Q_2 = ?$

Solution : $\frac{Q_2}{Q_1} = \frac{T_2}{T_1}$

$$\therefore \quad Q_2 = Q_1 \frac{T_2}{T_1} = 4200 \times \frac{350}{450}$$

$Q_2 = 3266.67\ J$

Efficiency, $\eta = 1 - \frac{T_2}{T_1}$

$$\eta = 1 - \frac{350}{450} = 0.2222 = 22.22\%$$

Work done

$W = Q_1 - Q_2 = 4200 - 3266.67$

$W = 933.33\ J$

8.7 A Carnot engine has the same efficiency, when operated

(i) between 100 K and 500 K

(ii) between T K and 900 K

Find the value of T

Solution : *(i) Here* $T_1 = 500\ K;\ T_2 = 100\ K$

$$\eta = 1 - \frac{T_2}{T_1} = 1 - \frac{100}{500} = 1 - 0.2 = 0.8$$

(ii) Now, $T_1 = 900\ K;\ T_2 = T$ *and* $\eta = 0.8$

Again, $\eta = 1 - \frac{T_2}{T_1}$

$0.8 = 1 - \frac{T}{900}$ *or* $\frac{T}{900} = 1 - 0.8 = 0.2$

$\therefore T = 180\ K$

8.8 In a refrigerator heat from inside at 277 K is transfered to a room at 300 K. How many joule of heat will be delivered to the room for each joule of electric energy consumed ideally?

Data : $T_1 = 300\ K$; $T_2 = 277\ K$
Solution : *COP of a refrigerator*

$$= \frac{T_2}{T_1 - T_2} = \frac{277}{300 - 277} = 12.04 \quad \text{...(1)}$$

Suppose for each joule of electric energy consumed an amount of heat Q_2 *is extracted from the inside of refrigerator. The amount of heat delivered to the room for each joule of electrical energy consumed is given by*

$$Q_1 = Q_2 + W = Q_2 + 1 \qquad (\because W = Q_1 - Q_2)$$

$$\therefore \qquad Q_1 - Q_2 = 1$$

Also for a refrigerator, $COP = \frac{Q_2}{Q_1 - Q_2} = Q_2$ ...(2)

From equations (1) and (2)

(i.e) $Q_2 = 12.04$

$\therefore Q_1 = Q_2 + 1 = 12.04 + 1 = 13.04\ J$

8.9 Two rods A and B of different material have equal length and equal temperature gradient. Each rod has its ends at temperatures T_1 and T_2. Find the condition under which rate of flow of heat through the rods A and B is same.

Solution : *Suppose the two rods A and B have the same temperature difference* $T_1 - T_2$ *across their ends and the length of each rod is l.*

When the two rods have the same rate of heat conduction,

$$\frac{K_1 A_1 (T_1 - T_2)}{l} = \frac{K_2 A_2 (T_1 - T_2)}{l}$$

$$K_1 A_1 = K_2 A_2 \text{ or } \frac{A_1}{A_2} = \frac{K_2}{K_1}$$

(i.e) for the same rate of heat conduction, the areas of cross - section of the two rods should be inversely proportional to their coefficients of thermal conductivity.

8.10 A metal cube takes 5 minutes to cool from 60° C to 52° C. How much time will it take to cool to 44° C, if the temperature of the surroundings is 32° C?

Solution : *While cooling from 60° C to 52° C*

Rate of cooling $= \frac{60-52}{5} = 1.6^o\ C/minute = \frac{1.6^o\ C}{60}$ *per second*

$\therefore$ *Average temperature while cooling* $= \frac{60+52}{2} = 56^o\ C$

$\therefore$ *Average temperature excess = 56 - 32 = 24° C*

According to Newton's law of cooling,

Rate of cooling α Temperature excess

$\therefore$ *Rate of cooling = K × temperature excess*

$$\frac{1.6}{60} = K \times 24 \qquad ...(1)$$

Suppose that the cube takes t seconds to cool from 52° C to 44° C

$$\therefore \text{ Rate of cooling} = \frac{52-44}{t} = \frac{8}{t}$$

Average temperature while cooling $= \frac{52+44}{2} = 48^o\ C$

$\therefore$ *Average temperature excess = 48 - 32 = 16° C*

According to Newton's law, Rate of cooling = K × (Temperature excess) $\frac{8}{t} = K \times 16$

Dividing equation (1) by equation (2)

$$\frac{1.6}{60} \times \frac{t}{8} = \frac{24}{16} \qquad = 450\ s$$